

# TRACE AND EXTENSION OPERATORS FOR BESOV SPACES AND TRIEBEL–LIZORKIN SPACES WITH VARIABLE EXPONENTS.

TAKAHIRO NOI

**ABSTRACT.** This paper is concerned with a boundedness of trace and extension operators for Besov spaces and Triebel–Lizorkin spaces on upper half space with variable exponents. To define trace and extension operators, we introduce a quarkonial decomposition for Besov spaces and Triebel–Lizorkin spaces with variable exponents on  $\mathbb{R}^n$ . Furthermore, we study trace and extension operators for Besov spaces and Triebel–Lizorkin spaces with variable exponents on upper half spaces  $\mathbb{R}_+^n$ .

**Keywords** Besov space, Triebel–Lizorkin space, variable exponents, quarkonial decomposition, trace operator, extension operator

**2010 Mathematics Subject Classification** Primary 42B35; Secondary 41A17.

## 1. INTRODUCTION

The function spaces with variable exponent(s) have a long history since the discovery by Orlicz[19] and in recent years, these spaces received great attention in connection with electrorheological fluids [21].

Besov spaces with variable exponents  $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  and Triebel–Lizorkin spaces with variable exponents  $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  were introduced by Almeida and Hästö [1] and Diening, Hästö and Roudenko [5], respectively. Diening, Hästö and Roudenko [5] proved the atomic decomposition for  $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  and applied the result to trace theorem. Kempka [12] proved the atomic, molecular and wavelet expansion for 2-microlocal Besov and Triebel–Lizorkin spaces with variable integrability. But, in the case of Besov space, the summability index  $q$  is a constant. Recently, Moura, Neves and Schneider [15] proved the boundedness of the trace operator for 2-microlocal Besov spaces by using atomic decomposition, but summability index  $q$  is a constant. Present author [16] studied a Fourier multiplier for  $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  and  $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  and the author and Sawano [18] studied atomic decomposition and complex interpolation for  $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  and  $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . Drihem [6] obtained a detailed atomic decomposition for  $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . Recently, the author and Izuki [17] studied a duality of  $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ ,  $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  and Herz spaces with variable exponents  $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ .

To prove a boundedness of the trace operator, we introduce quarkonial decompositions.

This paper concerns itself with quarkonial decompositions, trace operators and extension operators for Besov spaces and Triebel–Lizorkin spaces with variable exponents. First, we state atomic and quarkonial decompositions of Besov spaces and Triebel–Lizorkin spaces with

variable exponents. Secondly, we extend trace operators to Besov spaces and Triebel–Lizorkin spaces with variable exponents. Finally, we study trace and extension operators for Besov spaces and Triebel–Lizorkin spaces with variable exponents on upper half spaces  $\mathbb{R}_+^n$ .

## 2. DEFINITION OF BESOV SPACES AND TRIEBEL–LIZORKIN SPACES WITH VARIABLE EXPONENTS

Denote by  $\mathcal{P}_0(\mathbb{R}^n)$  the set of all measurable functions  $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$  such that

$$0 < p^- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x) = p^+ < \infty. \quad (1)$$

For  $p \in \mathcal{P}_0(\mathbb{R}^n)$ , let  $L^{p(\cdot)}(\mathbb{R}^n)$  be the set of measurable functions  $f$  on  $\mathbb{R}^n$  such that for some  $\lambda > 0$ ,

$$\int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1.$$

The infimum of such  $\lambda$  will be denoted by  $\|f\|_{L^{p(\cdot)}}(\mathbb{R}^n)$ . The set  $L^{p(\cdot)}(\mathbb{R}^n)$  becomes a quasi Banach function space equipped with the Luxemburg–Nakano norm  $\|f\|_{L^{p(\cdot)}}$ . More precisely,

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

If  $\Omega \subset \mathbb{R}^n$  is a measurable set, then we define

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

To define Besov and Triebel–Lizorkin spaces with variable exponents, we postulate the following conditions: There exists a positive constant  $C_{\log}(p)$  such that

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + |x - y|^{-1})} \quad (x, y \in \mathbb{R}^n, x \neq y) \quad (2)$$

and there exist a positive constant  $C_{\log}(p)$  and real number  $p_{\infty}$  such that

$$|p(x) - p_{\infty}| \leq \frac{C_{\log}(p)}{\log(e + |x|)} \quad (x \in \mathbb{R}^n). \quad (3)$$

The set of all real valued functions  $p : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying (2) and (3) is written by  $C^{\log}(\mathbb{R}^n)$ .

To define Besov spaces with variable exponents, we use mixed Lebesgue sequence space  $\ell^{q(\cdot)}(L^{p(\cdot)})$ .

Let  $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ . The space  $\ell^{q(\cdot)}(L^{p(\cdot)})$  is the collection of all sequences  $\{f_j\}_{j=0}^{\infty}$  of measurable functions on  $\mathbb{R}^n$  such that

$$\|\{f_j\}_{j=0}^{\infty}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf \left\{ \mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left( \left\{ \frac{f_j}{\mu} \right\}_{j=0}^{\infty} \right) \leq 1 \right\} < \infty,$$

where

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left( \{f_j\}_{j=0}^{\infty} \right) = \sum_{j=0}^{\infty} \inf \left\{ \lambda_j : \int_{\mathbb{R}^n} \left( \frac{|f_j(x)|}{\lambda_j^{\frac{1}{q(x)}}} \right)^{p(x)} dx \leq 1 \right\}.$$

Since we assume that  $q^+ < \infty$ ,

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left( \{f_j\}_{j=0}^{\infty} \right) = \sum_{j=0}^{\infty} \left\| |f_j|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}. \quad (4)$$

Almeida and Hästö [1] proved that  $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$  is a quasi-norm for all  $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and that  $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$  is a norm when  $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} \leq 1$ . Kempka and Vybíral [13] proved that  $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$  is a norm if  $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfy either  $1 \leq q(x) \leq p(x) \leq \infty$  almost everywhere on  $\mathbb{R}^n$  or  $\frac{1}{p(x)} + \frac{1}{q(x)} \leq 1$  for almost all  $x \in \mathbb{R}^n$ . Furthermore, they proved that there exist  $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfying  $\inf_{x \in \mathbb{R}^n} (p(\cdot), q(\cdot)) \geq 1$  such that a triangle inequality does not hold for  $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ . This means that  $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$  does not always become a norm even if  $p(\cdot)$  and  $q(\cdot)$  satisfy  $p^-, q^- \geq 1$ . However, we have following inequalities.

**Lemma 2.1.** (i) Let  $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ . Then

$$\|f + g\|_{L^{p(\cdot)}}^{\min(p^-, 1)} \leq \|f\|_{L^{p(\cdot)}}^{\min(p^-, 1)} + \|g\|_{L^{p(\cdot)}}^{\min(p^-, 1)}.$$

(ii) Let  $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ . Then

$$\|\{f_k + g_k\}_{k=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}^{\min(p^-, q^-, 1)} \leq \|\{f_k\}_{k=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}^{\min(p^-, q^-, 1)} + \|\{g_k\}_{k=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}^{\min(p^-, q^-, 1)}.$$

(iii) Let  $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$  and

$$\alpha = \min(q^-, 1) \min\left(1, \left(\frac{p}{q}\right)^-\right).$$

Then

$$\|\{f_k + g_k\}_{k=0}^\infty\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^\alpha \leq \|\{f_k\}_{k=0}^\infty\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^\alpha + \|\{g_k\}_{k=0}^\infty\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^\alpha.$$

*Proof.* Let  $r = \min(p^-, 1)$  and

$$\lambda_1 = \|f\|_{L^{p(\cdot)}}^{\min(p^-, 1)}, \quad \lambda_2 = \|g\|_{L^{p(\cdot)}}^{\min(p^-, 1)}.$$

Then we see that

$$\begin{aligned} & \int_{\mathbb{R}^n} \left( \frac{|f+g|}{(\lambda_1 + \lambda_2)^{1/r}} \right)^{p(x)} dx \\ &= \int_{\mathbb{R}^n} \left( \frac{|f+g|^r}{\lambda_1 + \lambda_2} \right)^{p(x)/r} dx \\ &\leq \int_{\mathbb{R}^n} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{|f|^r}{\lambda_1} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{|g|^r}{\lambda_2} \right)^{p(x)/r} dx \\ &\leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \int_{\mathbb{R}^n} \left( \frac{|f|^r}{\lambda_1} \right)^{p(x)/r} dx + \frac{\lambda_2}{\lambda_1 + \lambda_2} \int_{\mathbb{R}^n} \left( \frac{|g|^r}{\lambda_2} \right)^{p(x)/r} dx \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \int_{\mathbb{R}^n} \left( \frac{|f|}{\lambda_1^{1/r}} \right)^{p(x)} dx + \frac{\lambda_2}{\lambda_1 + \lambda_2} \int_{\mathbb{R}^n} \left( \frac{|g|}{\lambda_2^{1/r}} \right)^{p(x)} dx \\ &\leq 1. \end{aligned}$$

This implies (i).

Next we will prove (ii). Let  $r = \min(p^-, q^-, 1)$  and

$$\lambda_1 = \|\{f_k\}_{k=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}^{\min(p^-, q^-, 1)}, \quad \lambda_2 = \|\{g_k\}_{k=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}^{\min(p^-, q^-, 1)}.$$

Then we see that

$$\begin{aligned}
& \int_{\mathbb{R}^n} \left( \frac{\{\sum_{k=0}^{\infty} |f_k + g_k|^{q(x)}\}^{1/q(x)}}{(\lambda_1 + \lambda_2)^{1/r}} \right)^{p(x)} dx \\
& \leq \int_{\mathbb{R}^n} \left( \frac{\{\sum_{k=0}^{\infty} (|f_k|^r + |g_k|^r)^{q(x)/r}\}^{r/q(x)}}{\lambda_1 + \lambda_2} \right)^{p(x)/r} dx \\
& \leq \int_{\mathbb{R}^n} \left( \frac{\{\sum_{k=0}^{\infty} (|f_k|^r)^{q(x)/r}\}^{r/q(x)} + \{\sum_{k=0}^{\infty} (|g_k|^r)^{q(x)/r}\}^{r/q(x)}}{\lambda_1 + \lambda_2} \right)^{p(x)/r} dx \\
& \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \int_{\mathbb{R}^n} \left( \frac{\{\sum_{k=0}^{\infty} (|f_k|)^{q(x)}\}^{1/q(x)}}{\lambda_1^{1/r}} \right)^{p(x)} dx \\
& \quad + \frac{\lambda_2}{\lambda_1 + \lambda_2} \int_{\mathbb{R}^n} \left( \frac{\{\sum_{k=0}^{\infty} (|g_k|)^{q(x)}\}^{1/q(x)}}{\lambda_2^{1/r}} \right)^{p(x)} dx \\
& \leq 1.
\end{aligned}$$

This implies (ii).

Finally, we will prove (iii). Let

$$s = \min(q^-, 1), t = \min\left(1, \left(\frac{p}{q}\right)^-\right), \alpha = st$$

and

$$\lambda_1 = \|\{f_k\}_{k=0}^{\infty}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{\alpha}, \lambda_2 = \|\{g_k\}_{k=0}^{\infty}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{\alpha}.$$

Then we see that

$$\begin{aligned}
& \sum_{k=0}^{\infty} \left\| \left( \frac{|f_k + g_k|}{(\lambda_1 + \lambda_2)^{1/st}} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \\
& = \sum_{k=0}^{\infty} \left\| \left( \frac{|f_k + g_k|^{st}}{\lambda_1 + \lambda_2} \right)^{q(\cdot)/s} \right\|_{L^{\frac{p(\cdot)}{tq(\cdot)}}}^{1/t} \\
& = \sum_{k=0}^{\infty} \left\| \frac{\lambda_1}{\lambda_1 + \lambda_2} \left( \frac{|f_k|^{st}}{\lambda_1} \right)^{q(\cdot)/s} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \left( \frac{|g_k|^{st}}{\lambda_2} \right)^{q(\cdot)/s} \right\|_{L^{\frac{p(\cdot)}{tq(\cdot)}}}^{1/t} \\
& \leq \sum_{k=0}^{\infty} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \left\| \left( \frac{|f_k|^{st}}{\lambda_1} \right)^{q(\cdot)/s} \right\|_{L^{\frac{p(\cdot)}{tq(\cdot)}}} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \left\| \left( \frac{|g_k|^{st}}{\lambda_2} \right)^{q(\cdot)/s} \right\|_{L^{\frac{p(\cdot)}{tq(\cdot)}}} \right)^{1/t} \\
& \leq \sum_{k=0}^{\infty} \frac{\lambda_1}{\lambda_1 + \lambda_2} \left\| \left( \frac{|f_k|^{st}}{\lambda_1} \right)^{q(\cdot)/s} \right\|_{L^{\frac{p(\cdot)}{tq(\cdot)}}}^{1/t} + \sum_{k=0}^{\infty} \frac{\lambda_2}{\lambda_1 + \lambda_2} \left\| \left( \frac{|g_k|^{st}}{\lambda_2} \right)^{q(\cdot)/s} \right\|_{L^{\frac{p(\cdot)}{tq(\cdot)}}}^{1/t} \\
& = \sum_{k=0}^{\infty} \frac{\lambda_1}{\lambda_1 + \lambda_2} \left\| \left( \frac{|f_k|}{\lambda_1^{1/\alpha}} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} + \sum_{k=0}^{\infty} \frac{\lambda_2}{\lambda_1 + \lambda_2} \left\| \left( \frac{|g_k|}{\lambda_2^{1/\alpha}} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \\
& \leq 1.
\end{aligned}$$

Hence we have (iii).  $\square$

The set  $\Phi(\mathbb{R}^n)$  is the collection of all systems  $\theta = \{\theta_j\}_{j=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$  such that

$$\begin{cases} \text{supp } \theta_0 \subset \{x : |x| \leq 2\}, \\ \text{supp } \theta_j \subset \{x : 2^{j-1} \leq |x| \leq 2^{j+1}\} \text{ for } j = 1, 2, \dots, \end{cases}$$

for every multi-index  $\alpha$ , there exists a positive number  $c_\alpha$  such that

$$2^{j|\alpha|} |D^\alpha \theta_j(x)| \leq c_\alpha$$

for  $j = 0, 1, \dots$  and  $x \in \mathbb{R}^n$  and

$$\sum_{j=0}^{\infty} \theta_j(x) = 1$$

for  $x \in \mathbb{R}^n$ .

Let  $\theta$  be a continuous function on  $\mathbb{R}^n$  or the sum of finitely many characteristic functions of cubes in  $\mathbb{R}^n$ . Then  $\theta(D)$  is defined by  $\theta(D)f = \mathcal{F}^{-1}[\theta \cdot \mathcal{F}f]$ .

**Definition 2.2.** Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  and  $\alpha(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Let  $\theta = \{\theta_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$ . Besov space  $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$  with variable exponents is the collection of  $f \in \mathcal{S}'(\mathbb{R}^n)$  such that

$$\|f\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \left\| \left\{ 2^{j\alpha(\cdot)} \theta_j(D)f \right\}_{j=0}^{\infty} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Triebel–Lizorkin space  $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$  with variable exponents is the collection of  $f \in \mathcal{S}'(\mathbb{R}^n)$  such that

$$\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \left\| \left\{ 2^{j\alpha(\cdot)} \theta_j(D)f \right\}_{j=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty.$$

Here  $L^{p(\cdot)}(\ell^{q(\cdot)})$  is the space of all sequences  $\{g_j\}_0^\infty$  of measurable functions on  $\mathbb{R}^n$  such that quasi-norms

$$\|\{g_j\}_{j=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} = \left\| \left( \sum_{j=0}^{\infty} |g_j(\cdot)|^{q(\cdot)} \right)^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}} < \infty.$$

Let  $A_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$  be either  $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$  or  $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ .

**2.1. Fundamental results for variable exponents analysis.** Let  $A$  and  $B$  be positive constants or positive valued functions and  $c$  be a positive constant. In this paper, we use the following notations :

- If  $A \leq cB$  hold, then we write  $A \lesssim B$ .
- $A \gtrsim B$  means  $B \lesssim A$ .
- If  $A \lesssim B$  and  $B \lesssim A$ , then we write  $A \sim B$ .
- If there exists a constant  $c$  such that  $A = cB$ , then we write  $A \simeq B$ .

When we emphasize that the constant  $c$  as above is depend on some parameters  $\alpha, \beta, \gamma, \dots$ , then we use the following notations :

- We write  $A \lesssim_{\alpha, \beta, \gamma, \dots} B$  instead of  $A \lesssim B$ .
- We write  $A \gtrsim_{\alpha, \beta, \gamma, \dots} B$  instead of  $A \gtrsim B$ .
- We write  $A \sim_{\alpha, \beta, \gamma, \dots} B$  instead of  $A \sim B$ .
- We write  $A \simeq_{\alpha, \beta, \gamma, \dots} B$  instead of  $A \simeq B$ .

Similarly to classical theory, the following Hölder type inequalities [10, Theorem 2.3] hold.

**Theorem 2.3** ([10, Theorem 2.3]). *Let  $p_0(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$  with  $\frac{1}{p_0(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$ . Then we have  $\|fg\|_{L^{p_0(\cdot)}} \lesssim \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}}$  for any  $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$  and  $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$ .*

D. Cruz-Uribe et al. [2] proves the boundedness of classical operators, for example, singular integral operators and fractional integral operators on the space  $L^{p(\cdot)}(\mathbb{R}^n)$ . If  $f(\cdot)$  is a complex-valued locally Lebesgue-integrable function on  $\mathbb{R}^n$ , then

$$(\mathcal{M}f)(x) = \sup \frac{1}{|B|} \int_B |f(y)| dy$$

is called Hardy–Littlewood maximal operator, where the supremum is taken over all balls  $B$  centered at  $x$ . Furthermore, let  $0 < r \leq 1$ . If  $f(\cdot)$  is a complex-valued locally Lebesgue-integrable function on  $\mathbb{R}^n$ , then

$$(\mathcal{M}_r f)(x) = \left( \sup \frac{1}{|B|} \int_B |f(y)|^r dy \right)^{1/r}$$

is also called Hardy–Littlewood maximal operator. The next theorem is corresponding to the well-known maximal vector-valued inequality in the classical theory.

**Theorem 2.4** ([2, Corollary 2.1]). *If  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ , then, for all  $q \in (1, \infty)$ , there exists a constant  $c$  such that*

$$\|\{\mathcal{M}f_k\}_{k=0}^\infty\|_{L^{p(\cdot)}(\ell^q)} \leq c \|\{f_k\}_{k=0}^\infty\|_{L^{p(\cdot)}(\ell^q)} \quad (5)$$

for all sequences  $\{f_k\}_{k=0}^\infty \subset L^{p(\cdot)}(\mathbb{R}^n)$ .

It is well-known that (5) does not always hold if  $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  is not a constant function. However, Diening et al. [5] showed the following helpful theorem which takes the place of Theorem 2.4. Let

$$\eta_m(x) = (1 + |x|)^{-m} \quad \text{and} \quad \eta_{\nu,m}(x) = 2^{\nu n} \eta_m(2^\nu x)$$

for  $\nu \in \mathbb{N}_0$  and a positive real number  $m$ .

**Theorem 2.5** ([5, Theorem 3.2]). *Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n)$  with  $1 < p^- \leq p^+ < \infty$  and  $1 < q^- \leq q^+ < \infty$ . Then the inequality*

$$\|\{\eta_{k,m} * f_k\}_{k=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leq c \|\{f_k\}_{k=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$

holds for every sequence  $\{f_k\}_{k=0}^\infty$  of  $L^1_{\text{loc}}$ -functions and  $m > n$ .

Almeida et al. [1] showed the following helpful theorem for  $\ell^{q(\cdot)}(L^{p(\cdot)})$  quasi norm.

**Theorem 2.6** ([1, Lemma 4.7]). *Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n)$  with  $1 < p^- \leq p^+ < \infty$  and  $1 < q^- \leq q^+ < \infty$ . Then the inequality*

$$\|\{\eta_{k,m} * f_k\}_{k=0}^\infty\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \|\{f_k\}_{k=0}^\infty\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

holds for every sequence  $\{f_k\}_{k=0}^\infty$  of  $L^1_{\text{loc}}$ -functions and  $m > 2n$ .

**Remark 2.7.** Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$  with  $1 < p^- \leq p^+ < \infty$ . It is easy to see that the inequality

$$\|\{\eta_{k,m} * f_k\}_{k=0}^\infty\|_{\ell^\infty(L^{p(\cdot)})} \lesssim \|\{f_k\}_{k=0}^\infty\|_{\ell^\infty(L^{p(\cdot)})}$$

holds for every sequence  $\{f_k\}_{k=0}^\infty$  of  $L^1_{\text{loc}}$ -functions and  $m > n$  by Theorem 2.5.

By the proof of [5, Lemma 5.4], we see that the inequality

$$\|\{\eta_{k,m} * f_k\}_{k=0}^\infty\|_{L^{p(\cdot)}(\ell^\infty)} \lesssim \|\{f_k\}_{k=0}^\infty\|_{L^{p(\cdot)}(\ell^\infty)}$$

holds for every sequence  $\{f_k\}_{k=0}^\infty$  of  $L^1_{\text{loc}}$ -functions and  $m > 2n$ .

We often use the following relation between  $s(x)$  and  $s(y)$ .

**Lemma 2.8** ([5, Lemma 6.1]). *Let  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Then there exists a positive constant  $c$  such that*

$$2^{ks(x)} \eta_{k,2m}(x - y) \leq c 2^{ks(y)} \eta_{k,m}(x - y)$$

for all  $x, y \in \mathbb{R}^n$  and  $m > C_{\log}(s)$ .

**Lemma 2.9.** *Let  $r > 0$ ,  $\nu \in \mathbb{N}_0$  and  $m \geq n + 1$ . Then there exists  $c = c(r, m, n) > 0$  such that*

$$\frac{|f(x - z)|}{(1 + |2^\nu z|)^{\frac{m}{r}}} \leq c(\eta_{\nu, m} * |f|^r(x))^{\frac{1}{r}}$$

for all  $x \in \mathbb{R}^n$ ,  $z \in \mathbb{R}^n$  and  $f \in \mathcal{S}'(\mathbb{R}^n)$  with  $\text{supp } \mathcal{F}f \subset \{\xi : |\xi| \leq 2^{\nu+1}\}$ .

**Definition 2.10.** (i) Let  $\Omega$  be a compact subset of  $\mathbb{R}^n$ . Then  $\mathcal{S}^\Omega(\mathbb{R}^n)$  denotes the space of all elements  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  satisfying  $\text{supp } \mathcal{F}\varphi \subset \Omega$ .

(ii) Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ . For a sequence  $\Omega = \{\Omega_k\}_{k=0}^\infty$  of compact subsets of  $\mathbb{R}^n$ ,  $L_{p(\cdot)}^\Omega$  is the space of all sequences  $\{f_k\}_{k=0}^\infty$  of  $\mathcal{S}'(\mathbb{R}^n)$  such that

$$\text{supp } \mathcal{F}f_k \subset \Omega_k \quad (6)$$

and  $\|f_k\|_{L^{p(\cdot)}} < \infty$  for  $k = 0, 1, 2, \dots$ .

The author [16] proved the following Theorem.

**Theorem 2.11.** *Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  or  $q(\cdot) \equiv \infty$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Let  $\Omega = \{\Omega_k\}_{k=0}^\infty$  be a sequence of compact subsets of  $\mathbb{R}^n$  such that  $\Omega_k \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{k+1}\}$ .*

(i) *If  $v > \frac{n}{2} + \frac{4 \max\{n, C_{\log}(s)\}}{\min\{p^-, q^-\}}$ , then there exists a number  $c$  such that*

$$\|\{2^{ks(\cdot)} M_k(D) f_k\}_{k=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leq c \sup_l \|M_l(2^l \cdot)\|_{H_2^v} \|\{2^{ks(\cdot)} f_k\}_0^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$

for  $\{f_k(x)\}_{k=0}^\infty \in L_{p(\cdot)}^\Omega$  and  $\{M_k(x)\}_{k=0}^\infty \in H_2^v(\mathbb{R}^n)$ .

(ii) *If  $v > \frac{n}{2} + \frac{4 \max\{2n, C_{\log}(s)\}}{\min\{p^-, q^-\}}$ , then there exists a number  $c$  such that*

$$\|\{2^{ks(\cdot)} M_k(D) f_k\}_{k=0}^\infty\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq c \sup_l \|M_l(2^l \cdot)\|_{H_2^v} \|\{2^{ks(\cdot)} f_k\}_0^\infty\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

for  $\{f_k(x)\}_{k=0}^\infty \in L_{p(\cdot)}^\Omega$  and  $\{M_k(x)\}_{k=0}^\infty \in H_2^v(\mathbb{R}^n)$ .

Therefore, we obtain the lifting properties as a corollary of Theorem 2.11.

**Corollary 2.12** (Lifting properties). *Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Let  $\sigma \in \mathbb{R}$ ,  $k = 1, 2, \dots, m$  and  $m \in \mathbb{N}$ . Then*

$$\partial_k : A_{p(\cdot), q(\cdot)}^{s(\cdot)} \longrightarrow A_{p(\cdot), q(\cdot)}^{s(\cdot)-1}$$

is a continuous map. Furthermore, we have following properties:

- (1) The linear mapping  $(1 - \Delta)^\sigma$  is an isomorphism between  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}$  and  $A_{p(\cdot), q(\cdot)}^{s(\cdot)-2\sigma}$ .
- (2) The linear mapping  $(1 + (-\Delta)^m)$  is an isomorphism between  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}$  and  $A_{p(\cdot), q(\cdot)}^{s(\cdot)-2m}$ .
- (3) The linear mapping  $(1 + \partial_1^{4m} + \dots + \partial_n^{4m})$  is an isomorphism between  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}$  and  $A_{p(\cdot), q(\cdot)}^{s(\cdot)-4m}$ .

### 3. EMBEDDINGS FOR $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$

In this section, we deal with embeddings for  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ .

**Definition 3.1.** We define three linear spaces consist of bounded functions.

- (1) Denote by BC the linear space of all bounded continuous functions. Let  $f \in \text{BC}$ . Then we define  $\|f\|_{\text{BC}}$  such that  $\|f\|_{\text{BC}} := \|f\|_{L^\infty}$ .
- (2) Let  $m \in \mathbb{N}_0$ . Then, denote by  $\mathcal{B}^m$  the linear space of functions  $f : \mathbb{R}^n \longrightarrow \mathbb{C}$  such that  $f \in C^m$  and  $\partial^\alpha f \in \text{BC}$  for any multi-index  $\alpha$  with  $|\alpha| \leq m$ . We define the norm such that

$$\|f\|_{\mathcal{B}^m} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{\text{BC}}.$$

- (3) Denote by BUC the linear space consist of bounded uniformly continuous functions. Then we define  $\|f\|_{\text{BUC}}$  such that  $\|f\|_{\text{BUC}} := \|f\|_{L^\infty}$ .

Then the following embeddings are well-known.

**Lemma 3.2.** *For each  $m \in \mathbb{N}_0$ .  $B_{\infty,1}^m \hookrightarrow \mathcal{B}^m$  and  $B_{\infty,1}^0 \hookrightarrow \text{BUC}$  holds.*

Furthermore, we can prove the following embedding.

**Proposition 3.3.** *Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$  satisfy  $s(\cdot) > \frac{n}{p(\cdot)}$  and  $0 > \left(\frac{n}{p(\cdot)} - s(\cdot)\right)^+$ . Then  $A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n) \hookrightarrow \text{BUC}$ .*

To prove Proposition 3.3, we need Theorem 3.4.

**Theorem 3.4.** *Let  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$  and  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  satisfy  $0 < p(\cdot) \leq q(\cdot) < \infty$ . Let  $k > 0$  be a fixed number and  $B_{2^{j+k}} = \{x \in \mathbb{R}^n : |x| \leq 2^{j+k}\}$ . If  $\varphi_j \in \mathcal{S}^{B_{2^{j+k}}}(\mathbb{R}^n)$ , then there exists a positive number  $c$  such that*

$$\|2^{js(\cdot)} \varphi_j\|_{L^{q(\cdot)}} \leq c \left\| 2^{js(\cdot) + \frac{nj}{p(\cdot)} - \frac{nj}{q(\cdot)}} \varphi_j \right\|_{L^{p(\cdot)}}$$

and

$$\|2^{js(\cdot)} \varphi_j\|_{L^\infty} \leq c \left\| 2^{js(\cdot) + \frac{nj}{p(\cdot)}} \varphi_j \right\|_{L^{p(\cdot)}},$$

where  $c$  is independent of  $j$ .

We can prove Theorem 3.4 by an argument similar to proof of [16, Theorem 4.7]. So we omit the proof.

Now we prove Proposition 3.3.

*Proof of Proposition 3.3.* Let  $f \in A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . By Theorem 3.4, we have

$$\|\varphi_j(D)f\|_{L^\infty} \lesssim \left\| 2^{\frac{jn}{p(\cdot)}} \varphi_j(D)f \right\|_{L^{p(\cdot)}}.$$

Therefore, we see that

$$\sum_{j=0}^{\infty} \|\varphi_j(D)f\|_{L^\infty} \lesssim \sum_{j=0}^{\infty} \left\| 2^{\frac{jn}{p(\cdot)}} \varphi_j(D)f \right\|_{L^{p(\cdot)}} \lesssim \sum_{j=0}^{\infty} 2^{j\left(\frac{n}{p(\cdot)} - s(\cdot)\right)^+} \|f\|_{A_{p(\cdot),q(\cdot)}^{s(\cdot)}}.$$

This implies that  $A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n) \hookrightarrow B_{\infty,1}^0(\mathbb{R}^n) \hookrightarrow \text{BUC}$  by Lemma 3.2.  $\square$

Almeida and Hästö [1, Theorem 6.4] proved the Sobolev embedding for  $B_{p(\cdot),q(\cdot)}^{s(\cdot)}$ . We need a spacial case of Sobolev embeddings for  $F_{p(\cdot),q(\cdot)}^{s(\cdot)}$  and  $B_{p(\cdot),q(\cdot)}^{s(\cdot)}$ .

**Proposition 3.5.** *Let  $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Then we have*

$$A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n) \hookrightarrow B_{\infty,\infty}^{s(\cdot) - \frac{n}{p(\cdot)}}(\mathbb{R}^n).$$

*Proof.* As we mentioned above, Sobolev embedding for  $B_{p(\cdot),q(\cdot)}^{s(\cdot)}$  has been proved by Almeida and Hästö [1, Theorem 6.4]. Hence, we prove the  $F_{p(\cdot),q(\cdot)}^{s(\cdot)}$  case.

Let  $f \in F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . Without loss of generality, we can assume that  $\|f\|_{F_{p(\cdot),q(\cdot)}^{s(\cdot)}} = 1$ . By Theorem 3.4, we have

$$\left\| 2^{j\left(s(\cdot) - \frac{n}{p(\cdot)}\right)} \varphi_j(D)f \right\|_{L^\infty} \lesssim \left\| 2^{js(\cdot)} \varphi_j(D)f \right\|_{L^{p(\cdot)}} \lesssim 1.$$



By taking the  $\ell^\infty$  norm, we have  $\|f\|_{F_{p(\cdot),q(\cdot)}^{s(\cdot)}} \lesssim \|f\|_{B_{\infty,\infty}^{s(\cdot)-\frac{p}{p(\cdot)}}}$ .  $\square$

Almeida and Hästö [1] proved the following inclusion for  $A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ .

**Proposition 3.6** ([1, Theorem 6.1]). *Let  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ .*

(i) *Let  $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfy  $q_1(\cdot) \leq q_2(\cdot)$ . Then*

$$B_{p(\cdot),q_1(\cdot)}^{s(\cdot)}(\mathbb{R}^n) \subset B_{p(\cdot),q_2(\cdot)}^{s(\cdot)}(\mathbb{R}^n).$$

(ii) *Let  $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $s_1(\cdot), s_2(\cdot) \in C^{\log}(\mathbb{R}^n)$  such that  $\inf_{x \in \mathbb{R}^n} (s_1(x) - s_2(x)) > 0$ . Then*

$$B_{p(\cdot),q_1(\cdot)}^{s_1(\cdot)}(\mathbb{R}^n) \subset B_{p(\cdot),q_2(\cdot)}^{s_2(\cdot)}(\mathbb{R}^n).$$

(iii) *If  $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ , then*

$$B_{p(\cdot),\min\{p(\cdot),q(\cdot)\}}^{s(\cdot)}(\mathbb{R}^n) \subset F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n) \subset B_{p(\cdot),\max\{p(\cdot),q(\cdot)\}}^{s(\cdot)}(\mathbb{R}^n).$$

We have the counterpart of (i) and (ii) of Proposition 3.6 for  $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  by using similar argument of [1, Theorem 6.1].

**Proposition 3.7.** *Let  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ .*

(i) *Let  $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfy  $q_1(\cdot) \leq q_2(\cdot)$ . Then*

$$F_{p(\cdot),q_1(\cdot)}^{s(\cdot)}(\mathbb{R}^n) \subset F_{p(\cdot),q_2(\cdot)}^{s(\cdot)}(\mathbb{R}^n).$$

(ii) *Let  $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $s_1(\cdot), s_2(\cdot) \in C^{\log}(\mathbb{R}^n)$  such that  $\inf_{x \in \mathbb{R}^n} (s_1(x) - s_2(x)) > 0$ . Then*

$$F_{p(\cdot),q_1(\cdot)}^{s_1(\cdot)}(\mathbb{R}^n) \subset F_{p(\cdot),q_2(\cdot)}^{s_2(\cdot)}(\mathbb{R}^n).$$

#### 4. DECOMPOSITIONS OF BESOV AND TRIEBEL–LIZORKIN SPACES WITH VARIABLE EXPONENTS

In order to introduce a quarkonial decomposition for  $A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ , we use an atomic decomposition for  $A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . Kempka [12] proved the atomic decomposition for 2-microlocal Triebel–Lizorkin with variable exponents and 2-microlocal Besov spaces with variable exponents, but summability index  $q$  was constant in the Besov case. In a case that all exponents are variable exponents in Besov spaces, Drihem [6] proved the atomic decomposition for  $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . As we mentioned in Introduction, Moura, Neves and Schneider [15] proved the boundedness of the trace operator for 2-microlocal Besov spaces by using atomic decomposition, but summability index was constant. By using the results, we obtain the boundedness of the Trace operator for  $B_{p(\cdot),q}^{s(\cdot)}(\mathbb{R}^n)$  under the condition

$$s^- > \frac{1}{p^-} + (n-1) \left( \frac{1}{\min(1, p^-)} - 1 \right). \quad (7)$$

In this condition, it was considered essential infimum and essential supremum to each variable exponent  $p(\cdot)$  and  $s(\cdot)$ . However, as a condition of [5, Theorem 3.13] (although the theorem is Triebel–Lizorkin spaces with variable exponents case), we would like to consider a condition

where  $s^-$  and  $p^-$  are replaced by  $s(\cdot)$  and  $p(\cdot)$  respectively in (7). That is, we consider the boundedness of Trace operator for  $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  with the condition

$$\operatorname{ess\,inf}_{x \in \mathbb{R}^n} \left[ s(\cdot) - \left\{ \frac{1}{p(\cdot)} + (n-1) \left( \frac{1}{\min(1, p(\cdot), q(\cdot))} - 1 \right) \right\} \right] > 0, \quad (8)$$

which takes the place of the condition (7).

In the space  $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  case, as we mentioned in Introduction, Diening, Hästö and Roudenko [5] proved the boundedness of Trace operator ( Theorem 5.1 (i) ). In this paper, we consider the further results of boundedness of Trace operator ( Theorem 5.1 (ii) ).

To this end, we slightly change the definition of smooth atom which was introduced in Diening, Hästö and Roudenko [5] and a part of results on atomic decompositions by Kempka[12] and Drihem [6]. Therefore, the next subsection 4.1 essentially overlap with the works of Kempka [12] and Drihem [6].

**4.1. Atomic decomposition for Besov and Triebel–Lizorkin spaces with variable exponents.** We define  $\sigma_{p(\cdot),q(\cdot)}$  and  $\sigma_{p(\cdot)}$  such that

$$\begin{aligned} \sigma_{p(\cdot),q(\cdot)} &= n \left( \frac{1}{\min(1, p(\cdot), q(\cdot))} - 1 \right) \\ \sigma_{p(\cdot)} &= n \left( \frac{1}{\min(1, p(\cdot))} - 1 \right), \end{aligned}$$

where  $n$  is the spacial dimension.

Let  $\nu \in \mathbb{N}_0$  and  $m \in \mathbb{Z}^n$ . Then  $Q_{\nu,m} = \prod_{i=1}^n [2^{-\nu}m_i, 2^{-\nu}(m_i+1))$  and  $\chi_{\nu,m}$  is a characteristic function on  $Q_{\nu,m}$ .

**Definition 4.1.** Let  $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Let  $\nu \in \mathbb{N}_0$ ,  $m \in \mathbb{Z}^n$ . For double-index complex valued sequence  $\lambda = \{\lambda_{\nu,m}\}_{\nu,m}$ , we define

$$\|\lambda\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot)}} := \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \chi_{\nu,m} \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \quad (9)$$

$$\|\lambda\|_{f_{p(\cdot),q(\cdot)}^{s(\cdot)}} := \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \chi_{\nu,m} \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}. \quad (10)$$

We say that  $\lambda \in b_{p(\cdot),q(\cdot)}^{s(\cdot)}$  if (9) is finite and say that  $\lambda \in f_{p(\cdot),q(\cdot)}^{s(\cdot)}$  if (10) is finite.

Let  $a_{p(\cdot),q(\cdot)}^{s(\cdot)}$  be either  $b_{p(\cdot),q(\cdot)}^{s(\cdot)}$  or  $f_{p(\cdot),q(\cdot)}^{s(\cdot)}$ .

**Definition 4.2** (Atom). Let  $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Let  $K \in \mathbb{N}_0$ ,  $L \in \mathbb{Z}$ ,  $\nu \in \mathbb{N}_0$ ,  $m \in \mathbb{Z}^n$  and let  $\gamma > 1$ .

(1) A  $K$ -times continuous differentiable function  $a \in C^K(\mathbb{R}^n)$  is called  $[K, L]$ -atom centered at  $Q_{0,m}$ , if  $\operatorname{supp} a \subset \gamma Q_{0,m}$  and

$$\|\partial^\alpha a\|_\infty \leq 1, \quad |\alpha| \leq K. \quad (11)$$

(2) A  $K$ -times continuous differentiable function  $a \in C^K(\mathbb{R}^n)$  is called  $[K, L]$ -atom centered at  $Q_{\nu,m}$ , if  $\operatorname{supp} a \subset \gamma Q_{\nu,m}$ ,

$$\|\partial^\alpha a\|_\infty \leq 2^{\nu|\alpha|}, \quad |\alpha| \leq K \quad (12)$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x) dx = 0, \quad |\beta| \leq L. \quad (13)$$

The condition (13) is called moment condition. If  $L \leq -1$ , then no moment condition (13) required.

To prove the trace theorem, we need Theorem 4.7. We define the family of  $[K, L]$  smooth atoms.

**Definition 4.3.** Let  $K \in \mathbb{N}_0$  and  $L : \mathbb{R}^n \rightarrow \mathbb{R}$ . The family  $\{a_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  is said to be a family of  $[K, L]$  smooth atoms if  $a_{\nu,m}$  is a  $[K, \lfloor L_{Q_{\nu,m}}^- \rfloor]$  atom centered at  $Q_{\nu,m}$  for any  $\nu \in \mathbb{N}_0$  and  $m \in \mathbb{Z}^n$ . Here  $\lfloor A \rfloor = \max\{n \in \mathbb{Z} : n \leq A\}$  and  $L_{Q_{\nu,m}}^- = \text{ess inf}_{x \in Q_{\nu,m}} L(x)$ .

**Definition 4.4.** (1) We say that  $\{a_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  is a family of smooth atoms for  $F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  if it is a family of  $[K, L + \epsilon]$  smooth atoms, where  $K > s^+$  and

$$L(\cdot) = \sigma_{p(\cdot), q(\cdot)} - s(\cdot) \quad (14)$$

for some constant  $\epsilon > 0$ .

(2) We say that  $\{a_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  is a family of smooth atoms for  $B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  if it is a family of  $[K, L + \epsilon]$  smooth atoms, where  $K > s^+$  and

$$L(\cdot) = \sigma_{p(\cdot)} - s(\cdot)$$

for some constant  $\epsilon > 0$ .

**Remark 4.5.** (i) Drihem [6] proved the atomic decomposition for  $B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  under the moment condition  $L = \lfloor \sigma_{p^-} - s^- \rfloor$  and Kempka [12] proved the atomic decomposition for  $F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  under the moment condition  $L = \lfloor \sigma_{p^-, q^-} - s^- \rfloor$ .

(ii) A family of smooth atoms for  $F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  was introduced by Diening, Hästö and Roudenko [5]. Diening, Hästö and Roudenko [5] defined the smooth atoms with  $L(\cdot) = \sigma_{p(\cdot), q(\cdot)} - s(\cdot)$ . If  $\inf_{x \in \mathbb{R}^n} [s(x) - \sigma_{p(x), q(x)}] > 0$ , then we do not need the moment condition for the family of smooth atoms. For this reason, Diening, Hästö and Roudenko [5] proved the boundedness of Trace operator under the condition

$$\inf \left[ s(\cdot) - \left\{ \frac{1}{p(\cdot)} + (n-1) \left( \frac{1}{\min(1, p(\cdot))} - 1 \right) \right\} \right] > 0.$$

(iii) Let  $\{a_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  be a family of smooth atoms for  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . Then there exists a  $\epsilon > 0$  such that the atoms  $a_{\nu,m}$  are  $[K, L + 4\epsilon]$  smooth atoms, where  $L$  as in Definition 4.4. By the uniform continuity of  $p(\cdot)$ ,  $q(\cdot)$  and  $s(\cdot)$ , there exists a non negative integer  $\nu_0$  such that  $L_{Q_{\nu_0,m}}^- > (\sigma_{p(\cdot)})_{Q_{\nu_0,m}}^+ - s_{Q_{\nu_0,m}}^- - \epsilon$  and  $s_{Q_{\nu_0,m}}^- > s_{Q_{\nu_0,m}}^+ - \epsilon$  for any  $m \in \mathbb{Z}^n$ . Since  $p(\cdot)$ ,  $q(\cdot)$  and  $s(\cdot)$  also have a limit at infinity, there exists compact sets  $K \subset \mathbb{R}^n$  such that  $L_{\mathbb{R}^n \setminus K}^- > (\sigma_{p(\cdot)})_{\mathbb{R}^n \setminus K}^+ - s_{\mathbb{R}^n \setminus K}^- - \epsilon$  and  $s_{\mathbb{R}^n \setminus K}^- > s_{\mathbb{R}^n \setminus K}^+ - \epsilon$ . Then, since  $K \subset \mathbb{R}^n$  is a compact set, we can choose dyadic cubes  $\Omega_i$ ,  $i = 1, 2, \dots, R$ , of level  $\nu_0$  such that  $K \subset \cup_{i=1}^M \Omega_i$ . Furthermore we define  $\Omega_0 = \mathbb{R}^n \setminus \cup_{i=1}^R \Omega_i$ . These implies that  $L_{\Omega_i}^- > (\sigma_{p(\cdot)})_{\Omega_i}^+ - s_{\Omega_i}^- - \epsilon$  holds for any  $i = 0, 1, \dots, R$ . Note that if  $Q_{\nu,m} \subset \Omega_i$ , then

$$L_{Q_{\nu,m}}^- \geq L_{\Omega_i}^- \geq (\sigma_{p(\cdot)})_{\Omega_i}^+ - s_{\Omega_i}^- - \epsilon \geq (\sigma_{p(\cdot)})_{Q_{\nu,m}}^+ - s_{Q_{\nu,m}}^- - \epsilon$$

for  $B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  and

$$L_{Q_{\nu,m}}^- \geq L_{\Omega_i}^- \geq (\sigma_{p(\cdot), q(\cdot)})_{\Omega_i}^+ - s_{\Omega_i}^- - \epsilon \geq (\sigma_{p(\cdot), q(\cdot)})_{Q_{\nu,m}}^+ - s_{Q_{\nu,m}}^- - \epsilon$$

for  $F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . Hence, if  $Q_{\nu, m} \subset \Omega_i$ , then  $a_{\nu, m}$  is a  $[K, (\sigma_{p(\cdot)})_{\Omega_i}^+ - s_{\Omega_i}^- + 3\epsilon]$  smooth atom for  $B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  and is also a  $[K, (\sigma_{p(\cdot), q(\cdot)})_{\Omega_i}^+ - s_{\Omega_i}^- + 3\epsilon]$  smooth atom for  $F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ .

Let  $\{a_{\nu, m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  is a family of  $[K, L]$  smooth atoms. Then we need to check that  $f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m} a_{\nu, m}$  converges in  $\mathcal{S}'(\mathbb{R}^n)$ .

**Proposition 4.6.** *Let  $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . If  $\{a_{\nu, m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  is a family of smooth atoms for  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  and  $\lambda = \{\lambda_{\nu, m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in a_{p(\cdot), q(\cdot)}^{s(\cdot)}$ , then the sum*

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m} a_{\nu, m}$$

converges in  $\mathcal{S}'(\mathbb{R}^n)$ .

*Outline of the proof.* Let  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  arbitrary,  $\nu_0$  be as in Remark 4.5 and natural number  $k > \nu_0$ . Then we have

$$\left\langle \sum_{\nu=0}^k \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m} a_{\nu, m}, \varphi \right\rangle = \left\langle \sum_{\nu=0}^{\nu_0-1} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m} a_{\nu, m}, \varphi \right\rangle + \sum_{i=0}^R \left\langle \sum_{\nu=\nu_0}^k \sum_{\substack{m \in \mathbb{Z}^n: \\ Q_{\nu, m} \subset \Omega_i}} \lambda_{\nu, m} a_{\nu, m}, \varphi \right\rangle,$$

where  $\Omega_i$  and  $R$  are as in Remark 4.5 and the summation  $\sum_{\substack{m \in \mathbb{Z}^n: \\ Q_{\nu, m} \subset \Omega_i}}$  is taken over all  $m \in \mathbb{Z}^n$  such that  $Q_{\nu, m} \subset \Omega_i$ . Let fix non negative integer  $0 \leq i \leq R$ . We define

$$\lambda'_{\nu, m} = \begin{cases} \lambda_{\nu, m}, & \text{if } Q_{\nu, m} \subset \Omega_i \\ 0 & \text{otherwise} \end{cases}$$

and

$$a'_{\nu, m} = \begin{cases} a_{\nu, m}, & \text{if } Q_{\nu, m} \subset \Omega_i \\ 0 & \text{otherwise.} \end{cases}$$

Then  $a'_{\nu, m}$  is  $[K, (\sigma_{p(\cdot)})_{\Omega_i}^+ - s_{\Omega_i}^- + 3\epsilon]$  atom centered at  $Q_{\nu, m}$  by Remark 4.5 and  $L_{\Omega_i}^- + \epsilon > (\sigma_{p(\cdot)})_{\Omega_i}^+ - s_{\Omega_i}^-$  hold. Therefore, by using same argument of [12, Lemma 6] and [6, Theorem 3], we can prove

$$\sum_{\nu=\nu_0}^k \sum_{\substack{m \in \mathbb{Z}^n: \\ Q_{\nu, m} \subset \Omega_i}} \lambda_{\nu, m} a_{\nu, m} = \sum_{\nu=\nu_0}^k \sum_{m \in \mathbb{Z}^n} \lambda'_{\nu, m} a'_{\nu, m}$$

converges in  $\mathcal{S}'(\mathbb{R}^n)$  as  $k \rightarrow \infty$  for  $i = 0, 1, \dots, R$ . This implies that the sum

$$\sum_{\nu=0}^k \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m} a_{\nu, m} = \sum_{\nu=0}^{\nu_0-1} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m} a_{\nu, m} + \sum_{i=0}^R \sum_{\nu=\nu_0}^k \sum_{\substack{m \in \mathbb{Z}^n: \\ Q_{\nu, m} \subset \Omega_i}} \lambda_{\nu, m} a_{\nu, m}$$

convergence in  $\mathcal{S}'(\mathbb{R}^n)$  as  $k \rightarrow \infty$ . □

**Theorem 4.7.** *Let  $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Let  $\{a_{\nu, m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  is a family of smooth atoms for  $B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . If  $\lambda = \{\lambda_{\nu, m}\}_{(\nu, m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \in b_{p(\cdot), q(\cdot)}^{s(\cdot)}$ , then*

$$\left\| \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m} a_{\nu, m} \right\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot)}} \lesssim \|\lambda\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot)}}. \quad (15)$$

**Remark 4.8.** Theorem 4.7 is obtained by slightly changing a part of [6, Theorem 3].

To denote the outline of the proof of Theorem 4.7, we need following two Lemma.

**Lemma 4.9** ([6, Lemma 3]). *Let  $0 < a < 1$ ,  $0 < q \leq \infty$  and  $\delta > 0$  and let  $\{\epsilon_k\}_{k \in \mathbb{N}_0}$  be sequence of positive real numbers, such that*

$$\|\{\epsilon_k\}_{k \in \mathbb{N}_0}\|_{\ell^q} = I < \infty.$$

*The sequence  $\{\delta_k : \delta_k = \sum_{j=0}^{\infty} a^{|k-j|\delta} \epsilon_j\}_{k \in \mathbb{N}_0}$  is in  $\ell^q$  with*

$$\|\{\delta_k\}_{k \in \mathbb{N}_0}\|_{\ell^q} \leq cI,$$

*where  $c$  depends only on  $a$  and  $q$ .*

**Lemma 4.10** ([7, Lemma 3.3]). *Let  $\{\varphi_j\}$ ,  $j \in \mathbb{N}_0$  be a resolution of unity and let  $a_{\nu,m}$  be an  $[K, L]$ -atom. Then*

$$|\mathcal{F}^{-1} \varphi_j * a_{\nu,m}(x)| \lesssim \begin{cases} 2^{(\nu-j)K} (1 + 2^\nu |x - 2^{-\nu} m|)^{-M} & \text{if } \nu \leq j \\ 2^{(j-\nu)(L+n+1)} (1 + 2^j |x - 2^{-\nu} m|)^{-M} & \text{if } j \leq \nu, \end{cases}$$

*where  $M$  is sufficiently large.*

*Outline of the proof of Theorem 4.7.* Let  $f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}$ . Without loss of generality, we assume that  $\|\lambda\|_{b_{p(\cdot),q(\cdot)}^s(\cdot)} = 1$ . By using the similar argument of the proof of Theorem 3 of [6], it suffices to show that

$$\sum_{j=0}^{\infty} \left\| \left| c 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * f \right|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq C \quad \text{whenever} \quad \sum_{j=0}^{\infty} \left\| \left| 2^{js(\cdot)} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m} \right|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} = 1,$$

where  $\{\varphi_j\}_{j \in \mathbb{N}_0}$  is the resolution of unity as in Definition 2.2. Let  $0 < r < \max(1/q^+, p^-/q^+)$  and  $\nu_0$  as in Remark 4.5. Then we have

$$\begin{aligned} & \sum_{j=0}^{\infty} \left\| \left| c 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * f \right|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \\ & \leq \sum_{j=0}^{\infty} \left( \sum_{\nu=0}^{\nu_0-1} \left\| \left| c \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{\nu,m} a_{\nu,m})(\cdot) \right|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}} \right)^{\frac{1}{r}} \\ & \quad + \sum_{i=0}^R \sum_{j=0}^{\infty} \left( \sum_{\nu=\nu_0}^{\infty} \left\| \left| c \sum_{\substack{m \in \mathbb{Z}^n; \\ Q_{\nu,m} \subset \Omega_i}} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{\nu,m} a_{\nu,m})(\cdot) \right|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}(\Omega_i)}} \right)^{\frac{1}{r}} \\ & \leq I + \sum_{i=0}^R I_i, \end{aligned}$$

where

$$I_i = \sum_{j=0}^{\infty} \left( \sum_{\nu=\nu_0}^{\infty} \left\| \left| c \sum_{\substack{m \in \mathbb{Z}^n; \\ Q_{\nu,m} \subset \Omega_i}} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{\nu,m} a_{\nu,m})(\cdot) \right|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}(\Omega_i)}} \right)^{\frac{1}{r}}.$$

Firstly, we denote the outline of the proof of  $I_i \lesssim 1$  for any  $i = 0, 1, \dots, R$ . Let fix non negative integer  $0 \leq i \leq R$ . We define

$$\lambda'_{\nu,m} = \begin{cases} \lambda_{\nu,m}, & \text{if } Q_{\nu,m} \subset \Omega_i \\ 0 & \text{otherwise} \end{cases}$$

and

$$a'_{\nu,m} = \begin{cases} a_{\nu,m}, & \text{if } Q_{\nu,m} \subset \Omega_i \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$I_i = \sum_{j=0}^{\infty} \left( \sum_{\nu=\nu_0}^{\infty} \left\| \left| c \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda'_{\nu,m} a'_{\nu,m})(\cdot) \right|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}(\Omega_i)} \right)^{\frac{1}{r}}$$

and  $a'_{\nu,m}$  is  $[K, (\sigma_{p(\cdot)})_{\Omega_i}^+ - s_{\Omega_i}^- + 3\epsilon]$  atom centered at  $Q_{\nu,m}$  by Remark 4.5. By using same argument of the proof of [6, Theorem 3] with replacing  $L^{\frac{p(\cdot)}{rq(\cdot)}}$  by  $L^{\frac{p(\cdot)}{rq(\cdot)}}(\Omega_i)$ , we can prove  $I_i \lesssim 1$  for any  $i = 0, 1, \dots, R$ .

Finally, we denote the outline of the proof of  $I \lesssim 1$ . For any  $\nu \in \mathbb{N}_0$  and  $m \in \mathbb{Z}^n$ , it is easy to see that  $\lfloor L_{Q_{\nu,m}}^- \rfloor \geq \lfloor L_{\mathbb{R}^n}^- \rfloor$ . This implies that we can choose  $L$  as in Lemma 4.10 such that  $\lfloor L_{\mathbb{R}^n}^- \rfloor$ . By Lemma 4.10, we obtain

$$|2^{js(x)} \mathcal{F}^{-1} \varphi_j * a_{\nu,m}(x)| \leq \begin{cases} c 2^{-|j-\nu|(\lfloor L_{\mathbb{R}^n}^- \rfloor + 1 + n)} 2^{js(x)} (1 + 2^j |x - 2^{-\nu} m|)^{-M} & \text{if } j \leq \nu \\ c 2^{-|j-\nu|(K-s^+)} 2^{\nu s(x)} (1 + 2^\nu |x - 2^{-\nu} m|)^{-M} & \text{if } j \geq \nu. \end{cases} \quad (16)$$

Therefore, we have

$$\begin{aligned} & \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{\nu,m} a_{\nu,m}(\cdot) \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \\ & \leq \left\| \sum_{m \in \mathbb{Z}^n} c 2^{-|j-\nu|(\lfloor L_{\mathbb{R}^n}^- \rfloor + 1 + n)} 2^{js(\cdot)} \lambda_{\nu,m} \langle 2^j \cdot - 2^{j-\nu} m \rangle^{-M} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \end{aligned} \quad (17)$$

for  $j \leq \nu$  and

$$\begin{aligned} & \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{\nu,m} a_{\nu,m}(\cdot) \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \\ & \leq \left\| \sum_{m \in \mathbb{Z}^n} c 2^{(\nu-j)(K-s^+)} 2^{\nu s(\cdot)} \lambda_{\nu,m} \langle 2^\nu \cdot - m \rangle^{-M} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \end{aligned} \quad (18)$$

for  $j \geq \nu$ . Let  $0 < t < \min(1, p^-)$ . If there exists  $c > 0$  such that

$$\left\| c \sum_{m \in \mathbb{Z}^n} 2^{\nu s(\cdot)} \lambda_{\nu,m} \langle 2^\nu \cdot - m \rangle^{-M} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \leq \left\| \sum_{m \in \mathbb{Z}^n} 2^{\nu s(\cdot)} \lambda_{\nu,m} \chi_{\nu,m} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} + 2^{-\nu} \quad (19)$$

for  $j \geq \nu$  and

$$\begin{aligned} & \left\| c 2^{(j-\nu)(n/t-s^-)} \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \lambda_{\nu,m} \langle 2^\nu \cdot - m \rangle^{-M} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \\ & \leq \left\| \sum_{m \in \mathbb{Z}^n} 2^{\nu s(\cdot)} \lambda_{\nu,m} \chi_{\nu,m} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} + 2^{-j} \end{aligned} \quad (20)$$

for  $j \leq \nu$ , then

$$\begin{aligned} & \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{\nu, m} a_{\nu, m}(\cdot) \right\|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}} \\ & \leq 2^{(\nu-j)(K-s^+)rq^-} \left( \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{\nu s(\cdot)} \lambda_{\nu, m} \chi_{\nu, m} \right\|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}} + 2^{-\nu} \right) \end{aligned} \quad (21)$$

for  $j \geq \nu$  and

$$\begin{aligned} & \left\| \left\| c 2^{-|j-\nu|(\lfloor L_{\mathbb{R}^n}^- \rfloor + 1 + n)} \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \lambda_{\nu, m} \langle 2^\nu \cdot -m \rangle^{-M} \right\|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}} \\ & \leq 2^{(j-\nu)(\lfloor L^- \rfloor + n + 1 - n/t + s^-)rq^-} \left( \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{\nu s(\cdot)} \lambda_{\nu, m} \chi_{\nu, m} \right\|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}} + 2^{-j} \right) \\ & \leq C \left( \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{\nu s(\cdot)} \lambda_{\nu, m} \chi_{\nu, m} \right\|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}} + 2^{-j} \right) \end{aligned} \quad (22)$$

for  $j \leq \nu$ , where  $C = \max(1, 2^{(1-\nu_0)(\lfloor L^- \rfloor + n + 1 - n/t + s^-)q^-})$ . It is easy to see that

$$\begin{aligned} & c \sum_{j=0}^{\infty} \left\| \left\| \sum_{\nu=0}^{\nu_0-1} \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{\nu, m} a_{\nu, m})(\cdot) \right\|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{\frac{1}{r}} \\ & \leq c \sum_{j=0}^{\infty} \left( \sum_{\nu=0}^{\nu_0-1} \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{\nu, m} a_{\nu, m})(\cdot) \right\|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}} \right)^{\frac{1}{r}} \\ & \lesssim c \sum_{j=0}^{\infty} \sum_{\nu=0}^{\nu_0-1} \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{\nu, m} a_{\nu, m})(\cdot) \right\|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{\frac{1}{r}}. \end{aligned} \quad (23)$$

Let  $\nu = w$ . We estimate the right hand side of (23). We have

$$\begin{aligned} & \sum_{j=0}^{\infty} \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{w, m} a_{w, m})(\cdot) \right\|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{\frac{1}{r}} \\ & = \sum_{j=0}^w \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{w, m} a_{w, m})(\cdot) \right\|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{\frac{1}{r}} \\ & \quad + \sum_{j=w+1}^{\infty} \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{w, m} a_{w, m})(\cdot) \right\|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{\frac{1}{r}}. \end{aligned}$$

Then, by (21) and (22), we obtain

$$\begin{aligned}
& \sum_{j=0}^{\infty} \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{w,m} a_{w,m})(\cdot) \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \right\|^{\frac{1}{r}} \\
& \leq \sum_{j=0}^w C^{1/r} \left( \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{ws(\cdot)} \lambda_{w,m} \chi_{w,m} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \right\|^{\frac{1}{r}} + 2^{-j} \right) \\
& \quad + \sum_{j=w+1}^{\infty} 2^{(w-j)(K-s^+)q^-} \left( \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{ws(\cdot)} \lambda_{w,m} \chi_{w,m} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \right\|^{\frac{1}{r}} + 2^{-w} \right)^{\frac{1}{r}}.
\end{aligned}$$

By

$$\left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{ws(\cdot)} \lambda_{w,m} \chi_{w,m} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \right\|^{\frac{1}{r}} = \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{ws(\cdot)} \lambda_{w,m} \chi_{w,m} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}^{q(\cdot)} \right\| \leq 1,$$

we see that

$$\begin{aligned}
& \sum_{j=0}^{\infty} \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{w,m} a_{w,m})(\cdot) \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \right\|^{\frac{1}{r}} \\
& \leq (w+1) C^{1/r} 2^{1/r} + \sum_{j=w+1}^{\infty} 2^{|w-j|(s^+-K)q^-} \left( \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{ws(\cdot)} \lambda_{w,m} \chi_{w,m} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \right\|^{\frac{1}{r}} + 2^{-w} \right)^{\frac{1}{r}} \\
& \leq (w+1) C^{\frac{1}{r}} 2^{\frac{1}{r}} + \sum_{j=w+1}^{\infty} \left( \sum_{\nu=0}^w 2^{|\nu-j|(s^+-K)rq^-} \left( \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{\nu s(\cdot)} \lambda_{\nu,m} \chi_{\nu,m} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \right\|^{\frac{1}{r}} + 2^{-\nu} \right) \right)^{\frac{1}{r}} \\
& \leq (w+1) C^{\frac{1}{r}} 2^{\frac{1}{r}} + \sum_{j=w+1}^{\infty} \left( \sum_{\nu=0}^w 2^{-|\nu-j|(K-s^+)rq^-} \left( \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{\nu s(\cdot)} \lambda_{\nu,m} \chi_{\nu,m} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \right\|^{\frac{1}{r}} + 2^{-\nu} \right) \right)^{\frac{1}{r}}
\end{aligned}$$

Then, by Lemma 4.9, we obtain

$$\begin{aligned}
& \sum_{j=0}^{\infty} \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{w,m} a_{w,m})(\cdot) \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \right\|^{\frac{1}{r}} \\
& \leq (w+1) C^{\frac{1}{r}} 2^{\frac{1}{r}} + \sum_{j=w+1}^{\infty} \left( \sum_{\nu=0}^w 2^{-|\nu-j|(K-s^+)rq^-} \left( \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{\nu s(\cdot)} \lambda_{\nu,m} \chi_{\nu,m} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \right\|^{\frac{1}{r}} + 2^{-\nu} \right) \right)^{\frac{1}{r}} \\
& \leq (w+1) C^{1/r} 2^{1/r} + \sum_{j=0}^{\infty} \left( \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \lambda_{j,m} \chi_{j,m} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{rq(\cdot)} \right\|^{\frac{1}{r}} + 2^{-j} \right)^{\frac{1}{r}} \\
& \leq (w+1) C^{1/r} 2^{1/r} + D < \infty.
\end{aligned}$$



Therefore, we have

$$\begin{aligned} & c \sum_{j=0}^{\infty} \left\| \left| \sum_{\nu=0}^{\nu_0-1} \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} (\mathcal{F}^{-1} \varphi_j * \lambda_{\nu,m} a_{\nu,m})(\cdot) \right|^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}}^{\frac{1}{r}} \\ & \lesssim \left( \frac{\nu_0(\nu_0-1)}{2} + 1 \right) C^{1/r} 2^{1/r} + \nu_0 D < \infty \end{aligned}$$

by (23).

Therefore, we consider that (19) and (20). We can use similar argument of the proof of [6, Theorem 3], we have (19) and (20) because  $\nu < \nu_0 - 1$ .

□

By using similar arguments of the outline of the proof of Theorem 4.7 and similar arguments about atomic decompositions [12], we have the case of  $F_{p(\cdot),q(\cdot)}^{s(\cdot)}$ .

**Theorem 4.11.** *Let  $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Let  $\{a_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  is a family of smooth atoms for  $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . If  $\lambda = \{\lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \in f_{p(\cdot),q(\cdot)}^{s(\cdot)}$ , then*

$$\left\| \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m} \right\|_{F_{p(\cdot),q(\cdot)}^{s(\cdot)}} \lesssim \|\lambda\|_{f_{p(\cdot),q(\cdot)}^{s(\cdot)}}. \quad (24)$$

To denote the outline of the proof of Theorem 4.11, we need the following Theorem and Lemmas.

**Theorem 4.12** ([12, Corollary 5.6]). *Let  $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Furthermore let  $\{a_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  are  $[K, L]$  atoms for  $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ , where  $K > s^+$  and  $L = \lfloor \sigma_{p^-,q^-} - s^- \rfloor$ . If  $\lambda = \{\lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \in f_{p(\cdot),q(\cdot)}^{s(\cdot)}$ , then*

$$\left\| \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m} \right\|_{F_{p(\cdot),q(\cdot)}^{s(\cdot)}} \lesssim \|\lambda\|_{f_{p(\cdot),q(\cdot)}^{s(\cdot)}}. \quad (25)$$

**Lemma 4.13.** *Let  $0 < t < 1$ ,  $j, \nu \in \mathbb{N}_0$  and  $\{\lambda_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  be positive. Furthermore let  $M > 0$  be sufficiency large.*

(i) *Then*

$$\begin{aligned} & \sum_{m \in \mathbb{Z}^n} 2^{\nu s(x)} \lambda_{\nu,m} (1 + 2^j |x - 2^{-\nu} m|)^{-M} \\ & \leq c \max(1, 2^{(\nu-j)n/t}) \mathcal{M}_t \left( \sum_{m \in \mathbb{Z}^n} 2^{\nu s(\cdot)} \lambda_{\nu,m} \chi_{\nu,m}(\cdot) \right) (x) \end{aligned}$$

holds for any  $x \in \mathbb{R}^n$ .

(ii) *Then*

$$\begin{aligned} & \sum_{m \in \mathbb{Z}^n} 2^{\nu s(x)} \lambda_{\nu,m} (1 + 2^j |x - 2^{-\nu} m|)^{-M} \\ & \leq c 2^{h_n \alpha} \max(1, 2^{(\nu-j)n/t}) \left( \left[ \eta_{\nu,\alpha t} * \left( \sum_{m \in \mathbb{Z}^n} 2^{\nu t s(\cdot)} \lambda_{\nu,m} \chi_{\nu,m}^t(\cdot) \right) \right] (x) \right)^{1/t} \end{aligned}$$

holds for any  $x \in \mathbb{R}^n$  and for any positive real number  $\alpha > 0$ , where  $h_n$  is a positive number depend only on  $n$ .

*Proof.* (i) is proved in [6]. Hence we prove only (ii).

We use the argument similar to [6]. Let  $k \in \mathbb{N}_0$ . We define

$$\Omega_k = \left\{ m \in \mathbb{Z}^n : 2^{k-1} \leq 2^{\min(\nu, j)} |x - 2^{-\nu} m| \leq 2^k \right\}$$

and

$$\Omega_0 = \left\{ m \in \mathbb{Z}^n : 2^{\min(\nu, j)} |x - 2^{-\nu} m| \leq 1 \right\}.$$

Firstly we consider the case of  $\nu \leq j$ .

Let  $M = R + T$  and  $T > \frac{n}{t}$ . Then we obtain

$$\begin{aligned} \sum_{m \in \mathbb{Z}^n} 2^{\nu s(x)} \lambda_{\nu, m} (1 + 2^j |x - 2^{-\nu} m|)^{-M} &\leq \sum_{k=0}^{\infty} \sum_{m \in \Omega_k} 2^{\nu s(x)} \lambda_{\nu, m} (1 + 2^j |x - 2^{-\nu} m|)^{-M} \\ &\leq c \sum_{k=0}^{\infty} \sum_{m \in \Omega_k} 2^{\nu s(x)} \lambda_{\nu, m} 2^{-Mk} \\ &\leq c \sum_{k=0}^{\infty} 2^{-(T - \frac{n}{t})k} \sum_{m \in \Omega_k} 2^{\nu s(x)} \lambda_{\nu, m} 2^{-(R + \frac{n}{t})k} \\ &\leq c \left( \sup_{k \in \mathbb{N}_0} 2^{-(Rt+n)k} \sum_{m \in \Omega_k} 2^{\nu t s(x)} \lambda_{\nu, m}^t \right)^{\frac{1}{t}}. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\sum_{m \in \mathbb{Z}^n} 2^{\nu s(x)} \lambda_{\nu, m} (1 + 2^j |x - 2^{-\nu} m|)^{-M} \\ &\leq c \left( \sup_{k \in \mathbb{N}_0} 2^{-(Rt+n)k} \sum_{m \in \Omega_k} 2^{\nu t s(x)} \lambda_{\nu, m}^t \right)^{\frac{1}{t}} \\ &\leq c \left( \sup_{k \in \mathbb{N}_0} 2^{-Rtk + (\nu - k)n} \int_{\cup_{m \in \Omega_k} Q_{\nu, m}} \sum_{m \in \Omega_k} 2^{\nu t s(x)} \lambda_{\nu, m}^t \chi_{\nu, m}(y) dy \right)^{\frac{1}{t}}, \end{aligned} \quad (26)$$

where we use  $|\cup_{m \in \Omega_k} Q_{\nu, m}| \sim 2^{(k-\nu)n}$ . Using same argument in the proof of [6, Theorem 3], There exists a  $h_n \in \mathbb{N}_0$  such that  $|x - y| \leq 2^{k-\nu+h_n}$  for  $y \in \cup_{m \in \Omega_k} Q_{\nu, m}$ . This implies that  $y$  is located in some ball  $B(x, 2^{k-\nu+h_n})$  and that

$$1 \leq c \frac{2^{(k+h_n)\alpha t}}{(1 + 2^\nu |x - y|)^{\alpha t}}$$

holds for any  $\alpha > 0$ . Hence we see that

$$\begin{aligned} &\sum_{m \in \mathbb{Z}^n} 2^{\nu s(x)} \lambda_{\nu, m} (1 + 2^j |x - 2^{-\nu} m|)^{-M} \\ &\leq c \left( \sup_{k \in \mathbb{N}_0} 2^{-Rtk + (\nu - k)n} \int_{\cup_{m \in \Omega_k} Q_{\nu, m}} \sum_{m \in \Omega_k} 2^{\nu t s(x)} \lambda_{\nu, m}^t \chi_{\nu, m}(y) dy \right)^{\frac{1}{t}} \\ &\leq c \left( \sup_{k \in \mathbb{N}_0} 2^{-Rtk + (k+h_n)\alpha t - kn} \int_{B(x, 2^{k-\nu+h_n})} \sum_{m \in \Omega_k} \frac{2^{\nu n} 2^{\nu t s(x)} \lambda_{\nu, m}^t \chi_{\nu, m}(y)}{(1 + 2^\nu |x - y|)^{\alpha t}} dy \right)^{\frac{1}{t}}. \end{aligned}$$

Since  $s(\cdot) \in C_{\log}(\mathbb{R}^n)$ , we can prove that

$$2^{\nu s(x)} \leq c 2^{\beta k} 2^{\nu s(y)},$$

where  $\beta = \max(c_{\log}(s), s^+ - s^-)$ .

$$\begin{aligned} & \sum_{m \in \mathbb{Z}^n} 2^{\nu s(x)} \lambda_{\nu, m} (1 + 2^j |x - 2^{-\nu} m|)^{-M} \\ & \leq c \left( \sup_{k \in \mathbb{N}_0} 2^{-Rtk + (k+h_n)\alpha t - kn} \int_{B(x, 2^{k-\nu+h_n})} \sum_{m \in \Omega_k} \frac{2^{\nu n} 2^{\nu t s(x)} \lambda_{\nu, m}^t \chi_{\nu, m}(y)}{(1 + 2^\nu |x - y|)^{\alpha t}} dy \right)^{\frac{1}{t}} \\ & \leq c \left( \sup_{k \in \mathbb{N}_0} 2^{-(R+\alpha-\beta)kt} 2^{h_n \alpha t - kn} \left[ \eta_{\nu, \alpha t} * \left( \sum_{m \in \mathbb{Z}^n} 2^{\nu t s(y)} \lambda_{\nu, m}^t \chi_{\nu, m}(y) \right) \right] (x) \right)^{\frac{1}{t}}. \end{aligned}$$

Since  $R$  is sufficiently large such that

$$R > -\alpha + \max(c_{\log}(s), s^+ - s^-),$$

we get

$$\sum_{m \in \mathbb{Z}^n} 2^{\nu s(x)} \lambda_{\nu, m} (1 + 2^j |x - 2^{-\nu} m|)^{-M} \leq c 2^{h_n \alpha} \left[ \eta_{\nu, \alpha t} * \left( \sum_{m \in \mathbb{Z}^n} 2^{\nu t s(y)} \lambda_{\nu, m}^t \chi_{\nu, m}(y) \right) \right]^{\frac{1}{t}} (x).$$

Finally we consider the case of  $j \leq \nu$ . By using same argument as above, we have

$$\begin{aligned} & \sum_{m \in \mathbb{Z}^n} 2^{\nu s(x)} \lambda_{\nu, m} (1 + 2^j |x - 2^{-\nu} m|)^{-M} \\ & \leq c \left( \sup_{k \in \mathbb{N}_0} 2^{-(Rt+n)k} \sum_{m \in \Omega_k} 2^{\nu t s(x)} \lambda_{\nu, m}^t \right)^{\frac{1}{t}}. \end{aligned}$$

In the case of  $j \leq \nu$ , by  $|\cup_{m \in \Omega_k} Q_{\nu, m}| \sim 2^{(k-j)n}$ , we obtain

$$\begin{aligned} & \sum_{m \in \mathbb{Z}^n} 2^{\nu s(x)} \lambda_{\nu, m} (1 + 2^j |x - 2^{-\nu} m|)^{-M} \\ & \leq c \left( \sup_{k \in \mathbb{N}_0} 2^{-(Rt+n)k} \sum_{m \in \Omega_k} 2^{\nu t s(x)} \lambda_{\nu, m}^t \right)^{\frac{1}{t}} \\ & \leq c \left( \sup_{k \in \mathbb{N}_0} 2^{-Rtk + (j-k)n} \int_{\cup_{m \in \Omega_k} Q_{\nu, m}} \sum_{m \in \Omega_k} 2^{\nu t s(x)} \lambda_{\nu, m}^t \chi_{\nu, m}(y) dy \right)^{\frac{1}{t}}. \end{aligned}$$

This implies that

$$\begin{aligned} & \sum_{m \in \mathbb{Z}^n} 2^{\nu s(x)} \lambda_{\nu, m} (1 + 2^j |x - 2^{-\nu} m|)^{-M} \\ & \leq c \left( \sup_{k \in \mathbb{N}_0} 2^{-Rtk + (j-k)n} \int_{\cup_{m \in \Omega_k} Q_{\nu, m}} \sum_{m \in \Omega_k} 2^{\nu t s(x)} \lambda_{\nu, m}^t \chi_{\nu, m}(y) dy \right)^{\frac{1}{t}} \\ & \leq c 2^{(j-\nu)n/t} \left( \sup_{k \in \mathbb{N}_0} 2^{-Rtk + (\nu-k)n} \int_{\cup_{m \in \Omega_k} Q_{\nu, m}} \sum_{m \in \Omega_k} 2^{\nu t s(x)} \lambda_{\nu, m}^t \chi_{\nu, m}(y) dy \right)^{\frac{1}{t}}. \end{aligned}$$

Therefore, it is easy to see that

$$\sum_{m \in \mathbb{Z}^n} 2^{\nu s(x)} \lambda_{\nu, m} (1 + 2^j |x - 2^{-\nu} m|)^{-M} \leq c 2^{h_n \alpha} 2^{(\nu-j)n/t} \left( \left[ \eta_{\nu, \alpha t} * \left( \sum_{m \in \mathbb{Z}^n} 2^{\nu t s(\cdot)} \lambda_{\nu, m}^t \chi_{\nu, m}(\cdot) \right) \right] (x) \right)^{1/t}$$

by same argument as above.

□

**Lemma 4.14** ([11, Lemma 4.2]). Let  $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  with  $0 < q^- \leq q^+ < \infty$  and  $0 < q^- \leq q^+ < \infty$ . For any sequences  $\{g_j\}_{j=0}^\infty$  of nonnegative measurable functions on  $\mathbb{R}^n$  and  $\delta > 0$  let

$$G_j(x) = \sum_{k=0}^\infty 2^{-|k-j|\delta} g_k(x)$$

for all  $x \in \mathbb{R}^n$  and  $j \in \mathbb{N}_0$ . Then with constant  $c = c(p, q, \delta)$  we have

$$\|\{G_j\}_{j \in \mathbb{N}_0}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leq c \|\{g_j\}_{j \in \mathbb{N}_0}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}.$$

Now we prove Theorem 4.12.

As well as we mentioned in Remark 4.8, Theorem 4.11 is obtained by slightly changing a part of [12, Corollary 5.6] with the  $F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  case.

*Outline of the proof of Theorem 4.11.* Let  $f = \sum_{\nu=0}^\infty \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m} a_{\nu, m}$ . Without loss of generality, we assume that  $\|\lambda\|_{f^{s(\cdot)}_{p(\cdot), q(\cdot)}} = 1$ . Then we describe the outline of the proof of

$$\left\| \left\{ \left| 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * f \right|^r \right\}_{j=0}^\infty \right\|_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r})}^{1/r} \leq C,$$

where  $\{\varphi_j\}_{j \in \mathbb{N}_0}$  is the resolution of unity as in Definition 2.2. Let  $0 < r < \min(1, p^-)$ . By using similar arguments of the outline of the proof of Theorem 4.7, we consider the following inequality

$$\begin{aligned} & \left\| \left\{ \left| 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * f \right|^r \right\}_{j=0}^\infty \right\|_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r})}^{1/r} \\ & \leq \left\| \left\{ \left| 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \sum_{\nu=0}^{\nu_0-1} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m} a_{\nu, m} \right|^r \right\}_{j=0}^\infty \right\|_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r})}^{1/r} \\ & + \sum_{i=0}^R \left\| \left\{ \left| 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \sum_{\nu=\nu_0}^\infty \sum_{\substack{m \in \mathbb{Z}^n: \\ Q_{\nu, m} \subset \Omega_i}} \lambda_{\nu, m} a_{\nu, m} \right|^r \right\}_{j=0}^\infty \right\|_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r})(\Omega_i)}^{1/r} \\ & \leq I + \sum_{i=0}^R I_i, \end{aligned}$$

where

$$I_i = \left\| \left\{ \left| 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \sum_{\nu=\nu_0}^\infty \sum_{\substack{m \in \mathbb{Z}^n: \\ Q_{\nu, m} \subset \Omega_i}} \lambda_{\nu, m} a_{\nu, m} \right|^r \right\}_{j=0}^\infty \right\|_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r})(\Omega_i)}^{1/r}.$$

Then it suffices to show that  $I \lesssim 1$  and  $I_i \lesssim 1$  for any  $i = 0, 1, \dots, R$ .

Firstly, we denote the outline of the proof of  $I_i \lesssim 1$  for any  $i = 0, 1, \dots, R$ . Let fix non negative integer  $0 \leq i \leq R$ . We define

$$\lambda'_{\nu, m} = \begin{cases} \lambda_{\nu, m}, & \text{if } Q_{\nu, m} \subset \Omega_i \\ 0 & \text{otherwise} \end{cases}$$

and

$$a'_{\nu,m} = \begin{cases} a_{\nu,m}, & \text{if } Q_{\nu,m} \subset \Omega_i \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$I_i = \left\| \left\{ \left| 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda'_{\nu,m} a'_{\nu,m} \right|^r \right\}_{j=0}^{\infty} \right\|_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r}(\Omega_i))}^{1/r}$$

and  $a'_{\nu,m}$  is  $[K, (\sigma_{p(\cdot)}^+)_{\Omega_i} - s_{\Omega_i}^- + 3\epsilon]$  atom centered at  $Q_{\nu,m}$  by Remark 4.5. By using Theorem 4.12, we can prove  $I_i \lesssim 1$  for any  $i = 0, 1, \dots, R$ .

Finally, we denote the outline of the proof of  $I \lesssim 1$ . For any  $\nu \in \mathbb{N}_0$  and  $m \in \mathbb{Z}^n$ , it is easy to see that  $\lfloor L_{Q_{\nu,m}}^- \rfloor \geq \lfloor L_{\mathbb{R}^n}^- \rfloor$ . This implies that we can choose  $L$  as in Lemma 4.10 such that  $\lfloor L_{\mathbb{R}^n}^- \rfloor$ . By Lemma 4.10, we obtain

$$|2^{js(x)} \mathcal{F}^{-1} \varphi_j * a_{\nu,m}(x)| \leq \begin{cases} c 2^{-|j-\nu|(\lfloor L_{\mathbb{R}^n}^- \rfloor + 1 + n)} 2^{js(x)} (1 + 2^j |x - 2^{-\nu} m|)^{-M} & \text{if } j \leq \nu \\ c 2^{-|j-\nu|(K-s^+)} 2^{\nu s(x)} (1 + 2^\nu |x - 2^{-\nu} m|)^{-M} & \text{if } j \geq \nu, \end{cases} \quad (27)$$

where  $M$  is sufficiently large. Therefore, we have

$$\begin{aligned} & \left\| \left\{ \left| 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \sum_{\nu=0}^{\nu_0-1} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m} \right|^r \right\}_{j=0}^{\infty} \right\|_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r})}^{1/r} \\ & \leq \left\| \left\{ \sum_{j=0}^{\infty} \left( \left| \sum_{\nu=0}^{\nu_0-1} \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{\nu,m} a_{\nu,m} \right|^r \right)^{q(\cdot)/r} \right\}^{r/q(\cdot)} \right\|_{L^{p(\cdot)/r}}^{1/r}. \end{aligned} \quad (28)$$

Let  $\nu = w$ . Then

$$\begin{aligned} & \left\| \left\{ \sum_{j=0}^{\infty} \left( \left| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{w,m} a_{w,m} \right|^r \right)^{q(\cdot)/r} \right\}^{r/q(\cdot)} \right\|_{L^{p(\cdot)/r}}^{1/r} \\ & \lesssim \left( \sum_{j=0}^{w-1} \left\| \left\{ \left( \left| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{w,m} a_{w,m} \right|^r \right)^{q(\cdot)/r} \right\}^{r/q(\cdot)} \right\|_{L^{p(\cdot)/r}} \right)^{1/r} \\ & + \left( \sum_{j=w}^{\infty} \left\| \left\{ \left( \left| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{w,m} a_{w,m} \right|^r \right)^{q(\cdot)/r} \right\}^{r/q(\cdot)} \right\|_{L^{p(\cdot)/r}} \right)^{1/r} \\ & \lesssim \left( \sum_{j=0}^{w-1} \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{w,m} a_{w,m} \right\|_{L^{p(\cdot)/r}}^r \right\|_{L^{p(\cdot)/r}} \right)^{1/r} \\ & + \left( \sum_{j=w}^{\infty} \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{w,m} a_{w,m} \right\|_{L^{p(\cdot)/r}}^r \right\|_{L^{p(\cdot)/r}} \right)^{1/r} \\ & =: J_1 + J_2. \end{aligned} \quad (29)$$

We estimate  $J_1$ . By (27), we obtain

$$\begin{aligned}
J_1^r &= \sum_{j=0}^{w-1} \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{w,m} a_{w,m} \right\|_{L^{p(\cdot)/r}}^r \right\|_{L^{p(\cdot)}} \\
&\leq \sum_{j=0}^{w-1} \left\| \sum_{m \in \mathbb{Z}^n} 2^{-|j-w|(\lfloor L_{\mathbb{R}^n}^- \rfloor + 1 + n)} 2^{js(\cdot)} |\lambda_{w,m}| \langle 2^j \cdot - 2^{j-w} m \rangle^{-M} \right\|_{L^{p(\cdot)}}^r \\
&= \sum_{j=0}^{w-1} 2^{-|j-w|r(\lfloor L_{\mathbb{R}^n}^- \rfloor + 1 + n)} \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} |\lambda_{w,m}| \langle 2^j \cdot - 2^{j-w} m \rangle^{-M} \right\|_{L^{p(\cdot)}}^r. \tag{30}
\end{aligned}$$

Using similar arguments of the proof of [12, Theorem 3.13], we see that

$$\begin{aligned}
J_1^r &\leq \sum_{j=0}^{w-1} 2^{-|j-w|r(\lfloor L_{\mathbb{R}^n}^- \rfloor + 1 + n)} \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} |\lambda_{w,m}| \langle 2^j \cdot - 2^{j-w} m \rangle^{-M} \right\|_{L^{p(\cdot)}}^r \\
&\lesssim \sum_{j=0}^{w-1} 2^{-|j-w|r(\lfloor L_{\mathbb{R}^n}^- \rfloor + 1 + n + s^- - n/r)} \left\| \mathcal{M}_r \left( 2^{ws(y)} \sum_{m \in \mathbb{Z}^n} |\lambda_{w,m}| \chi_{w,m}(y) \right) (\cdot) \right\|_{L^{p(\cdot)}}^r
\end{aligned}$$

since we can take  $M$  sufficiently large and Lemma 4.13. It is well known that  $\mathcal{M}$  is bounded on  $L^{p(\cdot)/r}$  by [4, Theorem 4.3.8], we have

$$\begin{aligned}
J_1^r &\lesssim \sum_{j=0}^{w-1} 2^{-|j-w|r(\lfloor L_{\mathbb{R}^n}^- \rfloor + 1 + n + s^- - n/r)} \left\| \mathcal{M}_r \left( 2^{ws(y)} \sum_{m \in \mathbb{Z}^n} |\lambda_{w,m}| \chi_{w,m}(y) \right) (\cdot) \right\|_{L^{p(\cdot)}}^r \\
&\lesssim \sum_{j=0}^{w-1} 2^{-|j-w|r(\lfloor L_{\mathbb{R}^n}^- \rfloor + 1 + n + s^- - n/r)} \left\| 2^{ws(\cdot)} \sum_{m \in \mathbb{Z}^n} |\lambda_{w,m}| \chi_{w,m}(\cdot) \right\|_{L^{p(\cdot)}}^r \\
&\lesssim w \max \left( 1, 2^{wr(\lfloor L_{\mathbb{R}^n}^- \rfloor + 1 + n + s^- - n/r)} \right) \left\| 2^{ws(\cdot)} \sum_{m \in \mathbb{Z}^n} |\lambda_{w,m}| \chi_{w,m}(\cdot) \right\|_{L^{p(\cdot)}}^r.
\end{aligned}$$

By the assumption that  $\|\lambda\|_{f_{p(\cdot),q(\cdot)}^{s(\cdot)}} = 1$ , we obtain

$$\left\| 2^{ws(\cdot)} \sum_{m \in \mathbb{Z}^n} |\lambda_{w,m}| \chi_{w,m}(\cdot) \right\|_{L^{p(\cdot)}} \leq 1$$

because

$$\left\| 2^{ws(\cdot)} \sum_{m \in \mathbb{Z}^n} |\lambda_{w,m}| \chi_{w,m}(\cdot) \right\|_{L^{p(\cdot)}} \leq \left\| \left\{ \sum_{w=0}^{\infty} \left( 2^{ws(\cdot)} \sum_{m \in \mathbb{Z}^n} |\lambda_{w,m}| \chi_{w,m}(\cdot) \right)^{q(\cdot)} \right\}^{1/q(\cdot)} \right\|_{L^{p(\cdot)}}.$$

Therefore, we see that

$$J_1^r \lesssim w \max \left( 1, 2^{wr(\lfloor L_{\mathbb{R}^n}^- \rfloor + 1 + n + s^- - n/r)} \right). \tag{31}$$

By using similar calculation as above, we obtain

$$\begin{aligned}
J_2^r &= \sum_{j=w}^{\infty} \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{w,m} a_{w,m} \right\|_{L^{p(\cdot)/r}}^r \\
&\lesssim \sum_{j=w}^{\infty} \left\| \sum_{m \in \mathbb{Z}^n} 2^{(w-j)(K-s^+)} 2^{ws(\cdot)} \lambda_{w,m} \langle 2^w \cdot -m \rangle^{-M} \right\|_{L^{p(\cdot)}}^r \\
&\lesssim \sum_{j=w}^{\infty} \left\| 2^{(w-j)(K-s^+)} \mathcal{M}_r \left( 2^{ws(y)} \sum_{m \in \mathbb{Z}^n} \lambda_{w,m} \chi_{w,m}(y) \right) (\cdot) \right\|_{L^{p(\cdot)}}^r \\
&\lesssim \sum_{j=w}^{\infty} 2^{(w-j)r(K-s^+)} \left\| \mathcal{M}_r \left( 2^{ws(y)} \sum_{m \in \mathbb{Z}^n} \lambda_{w,m} \chi_{w,m}(y) \right) (\cdot) \right\|_{L^{p(\cdot)}}^r. \tag{32}
\end{aligned}$$

By the fact that  $\mathcal{M}_r$  is bounded on  $L^{p(\cdot)}$ , we have

$$\begin{aligned}
J_2^r &\lesssim \sum_{j=w}^{\infty} 2^{(w-j)r(K-s^+)} \left\| 2^{ws(\cdot)} \sum_{m \in \mathbb{Z}^n} \lambda_{w,m} \chi_{w,m} \right\|_{L^{p(\cdot)}}^r \\
&\lesssim \sum_{j=w}^{\infty} 2^{(w-j)r(K-s^+)}. \tag{33}
\end{aligned}$$

Since  $K > s^+$ , we have  $J_2^r < \infty$  for any  $0 < w < \nu_0 - 1$ . By (28), (31) and  $J_2^r < \infty$ , we see that

$$\begin{aligned}
&\left\| \left\{ 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \sum_{\nu=0}^{\nu_0-1} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m} \right\}_{j=0}^{\infty} \right\|_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r})}^r \\
&\leq \left\| \left\{ \sum_{j=0}^{\infty} \left( \sum_{\nu=0}^{\nu_0-1} \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{\nu,m} a_{\nu,m} \right\|_{L^{p(\cdot)/r}}^r \right)^{q(\cdot)/r} \right\}_{j=0}^{\infty} \right\|_{L^{p(\cdot)/r}}^{r/q(\cdot)} \\
&\lesssim \sum_{w=0}^{\nu_0-1} \left( \left\| \left\{ \sum_{j=0}^{w-1} \left( \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{\nu,m} a_{\nu,m} \right\|_{L^{p(\cdot)/r}}^r \right)^{q(\cdot)/r} \right\}_{j=0}^{w-1} \right\|_{L^{p(\cdot)/r}}^{r/q(\cdot)} \right. \\
&\quad \left. + \left\| \left\{ \sum_{j=w}^{\infty} \left( \left\| \sum_{m \in \mathbb{Z}^n} 2^{js(\cdot)} \mathcal{F}^{-1} \varphi_j * \lambda_{\nu,m} a_{\nu,m} \right\|_{L^{p(\cdot)/r}}^r \right)^{q(\cdot)/r} \right\}_{j=w}^{\infty} \right\|_{L^{p(\cdot)/r}}^{r/q(\cdot)} \right) \\
&\lesssim \sum_{w=0}^{\nu_0-1} (J_1^r + J_2^r) < \infty. \tag{34}
\end{aligned}$$

Therefore, we have  $I \lesssim \left( \sum_{w=0}^{\nu_0-1} (J_1^r + J_2^r) \right)^{1/r} < \infty$ .

□

**4.2. Quarkonial decomposition for Besov and Triebel–Lizorkin spaces with variable exponents : Regular case.** In this Section 4.2, we fix a function  $\psi \in \mathcal{S}(\mathbb{R}^n)$  uniquely such that

$$\sum_{m \in \mathbb{Z}^n} \psi(x - m) = 1$$

holds for any  $x \in \mathbb{R}^n$ . We also fix a number  $r > 0$  such that

$$\text{supp}(\psi) \subset B(2^r). \tag{35}$$

Here  $B(r) := \{y \in \mathbb{R}^n : |y| < r\}$ .

**Definition 4.15.** For a triple-index sequence  $\lambda = \{\lambda_{\nu,m}^\beta\}_{\beta \in \mathbb{N}_0^n, \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ , we define

$$\lambda^\beta := \{\lambda_{\nu,m}^\beta\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}, \quad \|\lambda\|_{a_{p(\cdot),q(\cdot),\rho}^{s(\cdot)}} := \sup_{\beta \in \mathbb{N}_0^n} 2^{\rho|\beta|} \|\lambda^\beta\|_{a_{p(\cdot),q(\cdot)}^{s(\cdot)}}. \quad (36)$$

**Definition 4.16.** Let  $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$  satisfy

$$\text{ess inf}_{x \in \mathbb{R}^n} \{s(\cdot) - \sigma_{p(\cdot),q(\cdot)}\} > 0 \quad (\text{Triebel--Lizorkin case}) \quad (37)$$

and

$$\text{ess inf}_{x \in \mathbb{R}^n} \{s(\cdot) - \sigma_{p(\cdot)}\} > 0 \quad (\text{Besov case}). \quad (38)$$

Let  $\beta \in \mathbb{N}_0^n$ ,  $\nu \in \mathbb{N}_0$  and  $m \in \mathbb{Z}^n$ . Then we define

$$\psi^\beta(x) := x^\beta \psi(x), \quad (\beta \text{qu})_{\nu,m}(x) := \psi^\beta(2^\nu x - m). \quad (39)$$

Furthermore we assume

$$\rho > r, \quad (40)$$

where  $r$  is as in (35).

Then we have a quarkonial decompositions for  $A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$

**Theorem 4.17** (Quarkonial decomposition of regular cases). *Let  $\rho$  be as in Definition 4.16 and let  $p(\cdot)$ ,  $q(\cdot)$  and  $s(\cdot)$  satisfy the condition of Definition 4.16. Let  $f \in \mathcal{S}'(\mathbb{R}^n)$ . Then  $f \in A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  if and only if there exists a triple-index sequence  $\lambda = \{\lambda_{\nu,m}^\beta\}_{\beta \in \mathbb{N}_0^n, \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  such that*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m}^\beta (\beta \text{qu})_{\nu,m} \quad (41)$$

and

$$\|\lambda\|_{a_{p(\cdot),q(\cdot),\rho}^{s(\cdot)}} < \infty. \quad (42)$$

Furthermore we can choose a coefficient  $\lambda$  such that

$$\|\lambda\|_{a_{p(\cdot),q(\cdot),\rho}^{s(\cdot)}} \sim \|f\|_{A_{p(\cdot),q(\cdot)}^{s(\cdot)}}. \quad (43)$$

**Remark 4.18.** In the classical setting, quarkonial decompositions of not only regular cases but also general cases for  $A_{p,q}^s(\mathbb{R}^n)$  are found in [24].

To prove Theorem 4.17, we need following Theorem 4.19, Corollary 4.20, Lemma 4.21 and Lemma 4.22.

**Theorem 4.19** (Fraizer--Jawerth  $\varphi$  transform). *Let  $\kappa \in \mathcal{S}(\mathbb{R}^n)$  satisfy  $\chi_{Q(3)} \leq \kappa \leq \chi_{Q(3+1/100)}$ , where  $Q(r) := \{y \in \mathbb{R}^n : \max(|y_1|, |y_2|, \dots, |y_n|) \leq r\}$ . Then*

$$f = (2\pi)^{-\frac{n}{2}} \sum_{m \in \mathbb{Z}^n} f\left(\frac{m}{R}\right) \mathcal{F}^{-1} \kappa(R \cdot - m) \quad (44)$$

holds for any  $f \in \mathcal{S}'(\mathbb{R}^n)$  such that  $\text{supp}(\mathcal{F}f) \subset Q(3R)$  ( $R > 0$ ).

Let  $\tau, \varphi \in \mathcal{S}(\mathbb{R}^n)$  such that

$$\chi_{B(2)} \leq \tau \leq \chi_{B(3)}, \quad \varphi_j(x) = \tau(2^{-j}x) - \tau(2^{-j+1}x), \quad j \in \mathbb{N}_0.$$

Since  $f = \tau(D)f + \sum_{\nu=1}^{\infty} \varphi_\nu(D)f$  for any  $f \in \mathcal{S}'(\mathbb{R}^n)$ , we have following Corollary.



**Corollary 4.20.** For any  $f \in \mathcal{S}'(\mathbb{R}^n)$ , we can write

$$f = (2\pi)^{-\frac{n}{2}} \sum_{m \in \mathbb{Z}^n} \tau(D) f(m) \mathcal{F}^{-1} \kappa(\cdot - m) \\ + (2\pi)^{-\frac{n}{2}} \sum_{\nu \in \mathbb{N}} \left( \sum_{m \in \mathbb{Z}^n} \varphi_\nu(D) f\left(\frac{m}{2^\nu}\right) \mathcal{F}^{-1} \kappa(2^\nu \cdot - m) \right).$$

**Lemma 4.21.** Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Furthermore, let  $0 < \alpha < \min(1, p^-, q^-)$  and  $\lambda^l := \{\lambda_{\nu, m+l}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  for any  $l \in \mathbb{Z}^n$ .

(1) If  $M > 2 \max(C_{\log}(s), 2n)$ , then

$$\|\lambda^l\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot)}} \lesssim \langle l \rangle^{\frac{M}{\alpha}} \|\lambda\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot)}} \quad (45)$$

holds, where  $\langle l \rangle := \sqrt{1 + l_1^2 + l_2^2 + \dots + l_n^2}$ .

(2) If  $M > 2 \max(C_{\log}(s), n)$ , then

$$\|\lambda^l\|_{f_{p(\cdot), q(\cdot)}^{s(\cdot)}} \lesssim \langle l \rangle^{\frac{M}{\alpha}} \|\lambda\|_{f_{p(\cdot), q(\cdot)}^{s(\cdot)}} \quad (46)$$

holds.

*Proof.* Let  $0 < \alpha < \min(p^-, q^-)$ . We fix  $\nu \in \mathbb{N}_0$  and  $x \in Q_{\nu, m}$ . Then  $Q_{\nu, m+l} \subset B(x, \sqrt{2n}2^{-\nu}\langle l \rangle)$  holds. Furthermore, we have

$$1 \lesssim \frac{((\sqrt{2n} + 1)\langle l \rangle)^M}{(1 + 2^\nu |x - y|)^M},$$

where  $y \in B(x, \sqrt{2n}2^{-\nu}\langle l \rangle)$ . Hence we see that

$$|\lambda_{\nu, m+l}|^\alpha \chi_{\nu, m}(x) \\ \lesssim \frac{\chi_{\nu, m}(x)}{|Q_{\nu, m+l}|} \int_{B(x, \sqrt{2n}2^{-\nu}\langle l \rangle)} \sum_{m' \in \mathbb{Z}^n} |\lambda_{\nu, m'}|^\alpha \chi_{\nu, m'}(y) dy \\ \lesssim ((\sqrt{2n} + 1)\langle l \rangle)^M \chi_{\nu, m}(x) \int_{B(x, \sqrt{2n}2^{-\nu}\langle l \rangle)} \frac{2^{\nu n}}{(1 + 2^\nu |x - y|)^M} \left( \sum_{m' \in \mathbb{Z}^n} |\lambda_{\nu, m'}|^\alpha \chi_{\nu, m'}(y) \right) dy \\ \lesssim ((\sqrt{2n} + 1)\langle l \rangle)^M \chi_{\nu, m}(x) \left( \eta_{\nu, M} * \left( \sum_{m' \in \mathbb{Z}^n} |\lambda_{\nu, m'}|^\alpha \chi_{\nu, m'} \right)^\alpha \right)(x).$$

By Lemma 2.8 and  $M \geq 2C_{\log}(s)$ , we have

$$\left| 2^{\nu s(x)} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m+l} \chi_{\nu, m}(x) \right| \lesssim \langle l \rangle^{\frac{M}{\alpha}} \left( \eta_{\nu, \frac{M}{2}} * \left( 2^{\nu s(\cdot)} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu, m}| \chi_{\nu, m}(\cdot) \right)^\alpha \right)^{\frac{1}{\alpha}}(x).$$

Thus (45) and (46) hold by Theorem 2.5 and 2.6.  $\square$

**Lemma 4.22.** Let  $\kappa \in \mathcal{S}(\mathbb{R}^n)$  satisfy  $\chi_{Q(3)} \leq \kappa \leq \chi_{Q(3+1/100)}$ . Then, for any  $N \gg 1$ , we have

$$|\partial^\alpha \mathcal{F}^{-1} \kappa(y)| \lesssim_N \langle \alpha \rangle^{2N} \langle y \rangle^{-2N}. \quad (47)$$

*Proof.* By using integration by parts, we can see that

$$\begin{aligned}\partial^\alpha \mathcal{F}^{-1} \kappa(y) &\simeq_n \int_{\mathbb{R}^n} (iz)^\alpha \kappa(z) \exp(iz \cdot y) \, dz \\ &\simeq_n \langle y \rangle^{-2N} \int_{\mathbb{R}^n} (iz)^\alpha \kappa(z) ((1 - \Delta_z)^N \exp(iz \cdot y)) \, dz \\ &\simeq_n \langle y \rangle^{-2N} \int_{\mathbb{R}^n} ((1 - \Delta_z)^N (iz)^\alpha \kappa(z)) \exp(iz \cdot y) \, dz,\end{aligned}$$

where  $\Delta_z = \sum_{i=1}^n \partial^2 / \partial z_i^2$ . Hence we have (47).  $\square$

From now, we prove Theorem 4.17.

*Proof of Theorem 4.17.* We divide the proof into the parts of sufficiency and necessity.

**Sufficiency.** Since we assume (40), we can take  $\epsilon > 0$  such that  $0 < \epsilon < \rho - r$ . For any  $\nu \in \mathbb{N}_0$  and  $m \in \mathbb{Z}^n$ , there exists  $d > 0$  such that  $\text{supp}(\beta \text{qu})_{\nu, m} \subset dQ_{\nu, m}$ . The conditions (37) and (38) imply that smooth atoms in  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  are not required to satisfy any moment conditions. Hence we can regard  $2^{-(r+\epsilon)|\beta|}(\beta \text{qu})_{\nu, m}$  as smooth atoms in  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . We define

$$f^\beta := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta (\beta \text{qu})_{\nu, m}.$$

By Theorem 4.7 and 4.11, we have

$$\|f^\beta\|_{A_{p(\cdot), q(\cdot)}^{s(\cdot)}} \lesssim 2^{-(\rho-r-\epsilon)|\beta|} \|\lambda\|_{a_{p(\cdot), q(\cdot), \rho}^{s(\cdot)}}.$$

Let

$$\sigma = \min \left\{ \min(q^-, 1) \min \left( 1, \left( \frac{p}{q} \right)^- \right), \min(p^-, q^-, 1) \right\}.$$

Since

$$\|f_1 + f_2\|_{A_{p(\cdot), q(\cdot)}^{s(\cdot)}}^\sigma \leq \|f_1\|_{A_{p(\cdot), q(\cdot)}^{s(\cdot)}}^\sigma + \|f_2\|_{A_{p(\cdot), q(\cdot)}^{s(\cdot)}}^\sigma$$

holds for any  $f_1, f_2 \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  by Lemma 2.1, we have  $f \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  and (42).

**Necessity.** Let  $f \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . Then we can write

$$\begin{aligned}f &= (2\pi)^{-\frac{n}{2}} \sum_{m \in \mathbb{Z}^n} \tau(D) f(m) \mathcal{F}^{-1} \kappa(\cdot - m) \\ &\quad + (2\pi)^{-\frac{n}{2}} \sum_{\nu \in \mathbb{N}} \left( \sum_{m \in \mathbb{Z}^n} \varphi_\nu(D) f\left(\frac{m}{2^\nu}\right) \mathcal{F}^{-1} \kappa(2^\nu \cdot - m) \right)\end{aligned}\tag{48}$$

by Corollary 4.20. For any  $(\nu, m) \in \mathbb{N}_0 \times \mathbb{Z}^n$ , we define

$$\Lambda_{\nu, m} = \begin{cases} \tau(D) f(m) & (\nu = 0) \\ \varphi_\nu(D) f\left(\frac{m}{2^\nu}\right) & (\nu \in \mathbb{N}). \end{cases}$$

Then we rewrite (48) to

$$f \simeq_n \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \Lambda_{\nu, m} \mathcal{F}^{-1} \kappa(2^\nu \cdot - m).\tag{49}$$

We can assume that  $\rho$  is a integer. By the Taylor expansion, we obtain

$$\begin{aligned} & \psi(2^{\nu+\rho}x - l)\mathcal{F}^{-1}\kappa(2^\nu x - m) \\ &= \sum_{\beta \in \mathbb{N}_0} \frac{\partial^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l - m)(2^\nu x - 2^{-\rho}l)^\beta \psi(2^{\nu+\rho}x - l)}{\beta!} \\ &= \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{-\rho|\beta|} \partial^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l - m) \psi^\beta(2^{\nu+\rho}x - l)}{\beta!}. \end{aligned}$$

Since  $\sum_{m \in \mathbb{Z}^n} \psi(x - m) = 1$ , we see that

$$\varphi_\nu(D)f \simeq_n \sum_{m \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{-\rho|\beta|}}{\beta!} \Lambda_{\nu,m} \partial^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l - m) \psi^\beta(2^{\nu+\rho}x - l). \quad (50)$$

Since we can regard (50) as converging in the topology of  $L^\infty$ , we can change the order of summation. Hence we can rewrite (50) as

$$\varphi_\nu(D)f \simeq_n \sum_{l \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \sum_{m \in \mathbb{Z}^n} \frac{2^{-\rho|\beta|}}{\beta!} \Lambda_{\nu,m} \partial^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l - m) (\beta \text{qu})_{\nu+\rho,l}(x).$$

Let

$$\lambda_{\nu+\rho,l}^\beta := \frac{2^{-\rho|\beta|}}{\beta!} \sum_{m \in \mathbb{Z}^n} \Lambda_{\nu,m} \partial^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l - m).$$

Then we have

$$f = \sum_{\nu \in \mathbb{N}_0} \varphi_\nu(D)f \simeq_n \sum_{\nu \in \mathbb{N}_0} \sum_{l \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \lambda_{\nu+\rho,l}^\beta (\beta \text{qu})_{\nu+\rho,l}. \quad (51)$$

Next we consider the  $a_{p(\cdot),q(\cdot),\rho}^{s(\cdot)}$  quasi norm of coefficients. Let  $l \in \mathbb{Z}^n$ ,  $l_0$  be a lattice point of  $[0, 2^\rho)^n$  and  $x \in Q_{\nu+\rho, 2^\rho l + l_0}$ . By (47), we obtain

$$\left| \lambda_{\nu+\rho, 2^\rho l + l_0}^\beta \right| \lesssim 2^{-\rho|\beta|} \sum_{m \in \mathbb{Z}^n} \langle l - m \rangle^{-N} |\Lambda_{\nu,m}| = c 2^{-\rho|\beta|} \sum_{m \in \mathbb{Z}^n} \langle m \rangle^{-N} |\Lambda_{\nu, m+l}|. \quad (52)$$

For each  $m \in \mathbb{Z}^n$ , we define

$$\eta_0 := \min \left( \min(1, p^-, q^-), \min(1, q^-) \min \left( 1, \left( \frac{p}{q} \right)^- \right) \right), \quad \Lambda^m := \{ |\Lambda_{\nu, m+l}| \}_{\nu \in \mathbb{N}_0, l \in \mathbb{Z}^n}.$$

By  $\mathbb{Z}^n = 2^\rho \mathbb{Z}^n + [0, 2^\rho)^n$ , (52) and Lemma 2.1, we see that

$$\|\lambda^\beta\|_{a_{p(\cdot),q(\cdot)}^{s(\cdot)}} \lesssim 2^{-\rho|\beta|} \left\| \sum_{m \in \mathbb{Z}^n} \langle m \rangle^{-N} \Lambda^m \right\|_{a_{p(\cdot),q(\cdot)}^{s(\cdot)}} \lesssim 2^{-\rho|\beta|} \left( \sum_{m \in \mathbb{Z}^n} \langle m \rangle^{-N\eta_0} \|\Lambda^m\|_{a_{p(\cdot),q(\cdot)}^{s(\cdot)}}^{\eta_0} \right)^{\frac{1}{\eta_0}}.$$

Since we can take  $N$  sufficiency large, we obtain

$$\|\lambda^\beta\|_{a_{p(\cdot),q(\cdot)}^{s(\cdot)}} \lesssim 2^{-\rho|\beta|} \left( \sum_{m \in \mathbb{Z}^n} \langle m \rangle^{(\frac{M}{\alpha} - N)\eta_0} \|\Lambda\|_{a_{p(\cdot),q(\cdot)}^{s(\cdot)}}^{\eta_0} \right)^{\frac{1}{\eta_0}} \lesssim 2^{-\rho|\beta|} \|\Lambda\|_{a_{p(\cdot),q(\cdot)}^{s(\cdot)}}$$

by Lemma 4.21, where  $\Lambda = \{\Lambda_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ . Hence we have

$$\|\lambda\|_{a_{p(\cdot),q(\cdot),\rho}^{s(\cdot)}} \lesssim \|\Lambda\|_{a_{p(\cdot),q(\cdot)}^{s(\cdot)}}$$

by  $\rho > r$ .

Finally, we prove  $\|\Lambda\|_{a_{p(\cdot),q(\cdot)}^{s(\cdot)}} \lesssim \|f\|_{A_{p(\cdot),q(\cdot)}^{s(\cdot)}}$ . Let  $\eta_1 = \min(1, p^-, q^-)$  and  $M$  be a sufficiency large. For any  $y \in Q_{\nu,m}$ , we have

$$\frac{1}{(1 + 2^\nu |y - 2^{-\nu} m|)^{\frac{2M}{\eta_1}}} \left| \varphi_\nu(D) f\left(\frac{m}{2^\nu}\right) \right| \lesssim \left( \eta_{\nu,M} * (\varphi_\nu(D) f)^{\frac{\eta_1}{2}}(y) \right)^{\frac{2}{\eta_1}}$$

by Lemma 2.9. Hence we see that

$$\begin{aligned} |\Lambda_{\nu,m}| &= \left| \varphi_\nu(D) f\left(\frac{m}{2^\nu}\right) \right| \\ &= (1 + 2^\nu |y - 2^{-\nu} m|)^{\frac{2M}{\eta_1}} \cdot \frac{1}{(1 + 2^\nu |y - 2^{-\nu} m|)^{\frac{2M}{\eta_1}}} \left| \varphi_\nu(D) f\left(\frac{m}{2^\nu}\right) \right| \\ &\lesssim (1 + n)^{\frac{2M}{\eta_1}} \frac{1}{(1 + 2^\nu |y - 2^{-\nu} m|)^{\frac{2M}{\eta_1}}} \left| \varphi_\nu(D) f\left(\frac{m}{2^\nu}\right) \right| \\ &\lesssim \left( \eta_{\nu,M} * (\varphi_\nu(D) f)^{\frac{\eta_1}{2}}(y) \right)^{\frac{2}{\eta_1}}. \end{aligned}$$

Since we have

$$|\Lambda_{\nu,m}| \lesssim \inf_{y \in Q_{\nu,m}} \left( \eta_{\nu,M} * (\varphi_\nu(D) f)^{\frac{\eta_1}{2}}(y) \right)^{\frac{2}{\eta_1}},$$

we obtain  $\|\Lambda\|_{a_{p(\cdot),q(\cdot)}^{s(\cdot)}} \lesssim \|f\|_{A_{p(\cdot),q(\cdot)}^{s(\cdot)}}$ . This proves the necessity of quarkonial decomposition.  $\square$

## 5. APPLICATION TO TRACE THEORY

Let  $n \geq 2$ . In this Section, we consider the Trace operator

$$\text{Tr}_{\mathbb{R}^n} : f(x', x_n) \mapsto f(x', 0), \quad x' \in \mathbb{R}^{n-1}, \quad f \in \mathcal{S}(\mathbb{R}^n). \quad (53)$$

We write  $x = (x', x_n) \in \mathbb{R}^n$ , where  $x' \in \mathbb{R}^{n-1}$  and  $x_n \in \mathbb{R}$ . Furthermore, we write  $\tilde{p}(x') = p(x', 0)$ ,  $\tilde{q}(x') = q(x', 0)$  and  $\tilde{s}(x') = s(x', 0)$ .

**Theorem 5.1.** *Assume that  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ .*

(1) *Let  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$  satisfy*

$$\text{ess inf}_{x \in \mathbb{R}^n} \left\{ s(\cdot) - \left[ \frac{1}{p(\cdot)} + (n-1) \left( \frac{1}{\min(1, p(\cdot))} - 1 \right) \right] \right\} > 0. \quad (54)$$

(a) *The operator  $\text{Tr}_{\mathbb{R}^n}$  can be extended as a surjective and continuous mapping from*

$$B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n) \text{ to } B_{\tilde{p}(\cdot),\tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1}).$$

(b) *The operator  $\text{Tr}_{\mathbb{R}^n}$  can be extended as a surjective and continuous mapping from*

$$F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n) \text{ to } F_{\tilde{p}(\cdot),\tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1}).$$

(2) *Let  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$  and  $k \in \mathbb{N}_0$  satisfy*

$$\text{ess inf}_{x \in \mathbb{R}^n} \left\{ s(\cdot) - \left[ k + \frac{1}{p(\cdot)} + (n-1) \left( \frac{1}{\min(1, p(\cdot))} - 1 \right) \right] \right\} > 0. \quad (55)$$

(a) *If  $g_0 \in B_{\tilde{p}(\cdot),\tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1})$ ,  $g_1 \in B_{\tilde{p}(\cdot),\tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)} - 1}(\mathbb{R}^{n-1})$ ,  $\dots$ ,  $g_k \in B_{\tilde{p}(\cdot),\tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)} - k}(\mathbb{R}^{n-1})$ , then there exists a  $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  such that  $\text{Tr}_{\mathbb{R}^n}(f) = g_0$ ,  $\text{Tr}_{\mathbb{R}^n}(\partial_{x_n} f) = g_1$ ,  $\dots$ ,  $\text{Tr}_{\mathbb{R}^n}(\partial_{x_n}^k f) = g_k$ .*

(b) *If  $g_0 \in F_{\tilde{p}(\cdot),\tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1})$ ,  $g_1 \in F_{\tilde{p}(\cdot),\tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)} - 1}(\mathbb{R}^{n-1})$ ,  $\dots$ ,  $g_k \in F_{\tilde{p}(\cdot),\tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)} - k}(\mathbb{R}^{n-1})$ , then there exists a  $f \in F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  such that  $\text{Tr}_{\mathbb{R}^n}(f) = g_0$ ,  $\text{Tr}_{\mathbb{R}^n}(\partial_{x_n} f) = g_1$ ,  $\dots$ ,  $\text{Tr}_{\mathbb{R}^n}(\partial_{x_n}^k f) = g_k$ .*

**Remark 5.2.** If  $p(\cdot) = p$ ,  $q(\cdot) = q$  and  $s(\cdot) = s$  are constant functions, then it is known that the assumption  $s > \frac{1}{p} + (n-1) \left( \frac{1}{\min(1,p)} - 1 \right)$  is optimal, see [23].

As we mentioned in Introduction, Diening, Hästö and Roudenko [5] proved Theorem 5.1-(1) for Triebel–Lizorkin spaces with variable exponents. In the case of Besov spaces with variable exponents, Moura, Neves and Schneider [15] proved Theorem 5.1-(1) for 2-microlocal Besov spaces with variable exponents, but summability index  $q$  was constant.

To prove Theorem 5.1, we need Lemma 5.3 and Lemma 5.6.

**Lemma 5.3** ([5, Lemma 7.1]). *Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ ,  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ ,  $\epsilon > 0$  and let  $\{E_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n}$  be a collection of sets such that  $E_{\nu,m} \subset 3Q_{\nu,m}$  and  $|E_{\nu,m}| \geq \epsilon |Q_{\nu,m}|$ . Then*

$$\left\| \{\lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \right\|_{f_{p(\cdot),q(\cdot)}^{s(\cdot)}} \sim \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu,m}| \chi_{E_{\nu,m}} \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \quad (56)$$

for any  $\{\lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \in f_{p(\cdot),q(\cdot)}^{s(\cdot)}$  and

$$\left\| \{\lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \right\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot)}} \sim \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu,m}| \chi_{E_{\nu,m}} \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \quad (57)$$

for any  $\{\lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \in b_{p(\cdot),q(\cdot)}^{s(\cdot)}$ .

Triebel–Lizorkin case (56) is proved in [5, Lemma 7.1]. By using same argument of the proof of [5, Lemma 7.1], we can prove Besov case (57).

**Lemma 5.4** ([5, Lemma 7.2]). *Let  $p_1(\cdot), p_2(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  and  $s_1(\cdot), s_2(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Assume that  $p_1(\cdot) = p_2(\cdot)$ ,  $q_1(\cdot) = q_2(\cdot)$  and  $s_1(\cdot) = s_2(\cdot)$  in the upper or lower half space. For double-index complex-valued sequence  $\lambda = \{\lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n}$ ,*

$$\left\| \{\delta_{m_n,0} \lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \right\|_{a_{p_1(\cdot),q_1(\cdot)}^{s_1(\cdot)}(\mathbb{R}^n)} \sim \left\| \{\delta_{m_n,0} \lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \right\|_{a_{p_2(\cdot),q_2(\cdot)}^{s_2(\cdot)}(\mathbb{R}^n)}, \quad (58)$$

where

$$\delta_{m_n,0} = \begin{cases} 1 & m_n = 0, \\ 0 & m_n \neq 0. \end{cases}$$

*Proof.* Triebel–Lizorkin case is proved in [5]. Hence we prove the Besov case by using similar argument in the proof of [5, Lemma 7.2]. We prove the case of  $p_1(\cdot) = p_2(\cdot)$ ,  $q_1(\cdot) = q_2(\cdot)$  and  $s_1(\cdot) = s_2(\cdot)$  in the lower half space because it is obvious that (58) holds if  $p_1(\cdot) = p_2(\cdot)$ ,  $q_1(\cdot) = q_2(\cdot)$  and  $s_1(\cdot) = s_2(\cdot)$  in the upper half space.

For  $m = (m', 0) \in \mathbb{Z}^n$ , we put

$$E_{\nu,m} = \left\{ (x', x_n) \in \mathbb{R}^n : (x', -x_n) \in Q_{\nu,m}, -\frac{3}{4}2^{-\nu} \leq x_n \leq -\frac{1}{2}2^{-\nu} \right\};$$

for all other  $m \in \mathbb{Z}^n$ , we put  $E_{\nu,m} = Q_{\nu,m}$ . Since  $E_{\nu,m}$  is supported in the lower space when  $m_n = 0$ , by Lemma 5.3, we see that

$$\begin{aligned} \left\| \{\delta_{m_n,0} \lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \right\|_{b_{p_2(\cdot), q_2(\cdot)}^{s_2(\cdot)}(\mathbb{R}^n)} &\sim \left\| \left\{ 2^{\nu s_2(\cdot)} \sum_{m \in \mathbb{Z}^n} \delta_{m_n,0} \lambda_{\nu,m} E_{\nu,m} \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q_2(\cdot)}(L^{p_2(\cdot)})} \\ &\sim \left\| \left\{ 2^{\nu s_1(\cdot)} \sum_{m \in \mathbb{Z}^n} \delta_{m_n,0} \lambda_{\nu,m} E_{\nu,m} \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q_1(\cdot)}(L^{p_1(\cdot)})} \\ &\sim \left\| \{\delta_{m_n,0} \lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \right\|_{b_{p_1(\cdot), q_1(\cdot)}^{s_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

This complete the proof.  $\square$

**Corollary 5.5** (cf [5, Proposition 7.3]). *Let  $p_1(\cdot), p_2(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  and  $s_1(\cdot), s_2(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Assume that  $p_1(x) = p_2(x)$ ,  $q_1(x) = q_2(x)$  and  $s_1(x) = s_2(x)$  for all  $x \in \mathbb{R}^{n-1} \times \{0\}$ . For double-index complex-valued sequence  $\lambda = \{\lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n}$ ,*

$$\left\| \{\delta_{m_n,0} \lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \right\|_{a_{p_1(\cdot), q_1(\cdot)}^{s_1(\cdot)}(\mathbb{R}^n)} \sim \left\| \{\delta_{m_n,0} \lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \right\|_{a_{p_2(\cdot), q_2(\cdot)}^{s_2(\cdot)}(\mathbb{R}^n)}. \quad (59)$$

Triebel–Lizorkin case is proved in [5, Proposition 7.3]. By using same argument in the proof of [5, Proposition 7.3], we can prove the Besov case.

**Lemma 5.6.** *Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . For double-index complex-valued sequence  $\lambda = \{\lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n}$ , we have*

$$\left\| \{\lambda_{\nu,(m',0)}\}_{(\nu,m') \in \mathbb{N}_0 \times \mathbb{Z}^{n-1}} \right\|_{f_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1})} \sim \left\| \{\delta_{m_n,0} \lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \right\|_{f_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)}.$$

*Proof.* By Corollary 5.5, it suffices to consider the case  $p(\cdot)$ ,  $q(\cdot)$  and  $s(\cdot)$  independent of the  $n$ -th coordinate for  $|x_n| \leq 2$ . Let  $\tilde{Q}_{\nu,m'} = Q_{\nu,m'} \times [2^{-\nu}, 2^{-\nu+1})$  for  $\nu \in \mathbb{N}_0$  and  $m' \in \mathbb{Z}^{n-1}$ . Firstly, we prove

$$\begin{aligned} &\left\| \left\{ 2^{\nu(\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)})} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu,(m',0)} \chi_{\nu,m'} \right\}_{\nu=0}^{\infty} \right\|_{L^{\tilde{p}(\cdot)}(\ell^{\tilde{p}(\cdot)})} \\ &\sim \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu,(m',0)} \chi_{\tilde{Q}_{\nu,m'}} \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}. \end{aligned} \quad (60)$$

Let  $\lambda > 0$ . By  $\tilde{Q}_{\nu, m'} = Q_{\nu, m'} \times [2^{-\nu}, 2^{-\nu+1})$ , we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^{n-1}} \left| \frac{\left\{ \sum_{\nu \in \mathbb{N}_0} \left( 2^{\nu(s(x',0) - \frac{1}{p(x',0)})} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\nu, m'} \right)^{p(x',0)} \right\}^{\frac{1}{p(x',0)}}}{\lambda} \right|^{p(x',0)} dx' \\
&= \int_{\mathbb{R}^{n-1}} \sum_{\nu \in \mathbb{N}_0} 2^{\nu p(x',0)s(x',0) - \nu} \left| \frac{\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\nu, m'}(x')}{\lambda} \right|^{p(x',0)} dx' \\
&= \int_{\mathbb{R}^{n-1}} \sum_{\nu \in \mathbb{N}_0} 2^{\nu p(x',0)s(x',0)} \left( \int_{2^{-\nu}}^{2^{-\nu+1}} \left| \frac{\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\nu, m'}(x')}{\lambda} \right|^{p(x',0)} \chi_{[2^{-\nu}, 2^{-\nu+1})}(x_n) dx_n \right) dx' \\
&= \int_{\mathbb{R}^{n-1}} \sum_{\nu \in \mathbb{N}_0} 2^{\nu p(x',0)s(x',0)} \left( \int_{2^{-\nu}}^{2^{-\nu+1}} \left| \frac{\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\nu, m' \times [2^{-\nu}, 2^{-\nu+1})}(x', x_n)}{\lambda} \right|^{p(x',0)} dx_n \right) dx' \\
&= \int_{\mathbb{R}^{n-1}} \sum_{\nu \in \mathbb{N}_0} 2^{\nu p(x',0)s(x',0)} \left( \int_{2^{-\nu}}^{2^{-\nu+1}} \left| \frac{\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x', x_n)}{\lambda} \right|^{p(x',0)} dx_n \right) dx'. \quad (61)
\end{aligned}$$

By assumptions for  $p(\cdot)$ ,  $q(\cdot)$  and  $s(\cdot)$ , we see that

$$\begin{aligned}
& \sum_{\nu \in \mathbb{N}_0} 2^{\nu p(x',0)s(x',0)} \left( \int_{2^{-\nu}}^{2^{-\nu+1}} \left| \frac{\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x', x_n)}{\lambda} \right|^{p(x',0)} dx_n \right) \\
&= \sum_{\nu \in \mathbb{N}_0} \left( \int_{2^{-\nu}}^{2^{-\nu+1}} 2^{\nu p(x',0)s(x',0)} \left| \frac{\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x', x_n)}{\lambda} \right|^{p(x',0)} dx_n \right) \\
&= \sum_{\nu \in \mathbb{N}_0} \left( \int_{2^{-\nu}}^{2^{-\nu+1}} 2^{\nu p(x', x_n)s(x', x_n)} \left| \frac{\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x', x_n)}{\lambda} \right|^{p(x', x_n)} dx_n \right) \\
&= \sum_{\nu \in \mathbb{N}_0} \left( \int_{-\infty}^{\infty} 2^{\nu p(x', x_n)s(x', x_n)} \left| \frac{\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x', x_n)}{\lambda} \right|^{p(x', x_n)} dx_n \right) \\
&= \int_{-\infty}^{\infty} \sum_{\nu \in \mathbb{N}_0} 2^{\nu p(x', x_n)s(x', x_n)} \left| \frac{\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x', x_n)}{\lambda} \right|^{p(x', x_n)} dx_n. \quad (62)
\end{aligned}$$

By (61) and (62), we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^{n-1}} \left| \frac{\left\{ \sum_{\nu \in \mathbb{N}_0} \left( 2^{\nu(s(x',0) - \frac{1}{p(x',0)})} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\nu, m'} \right)^{p(x',0)} \right\}^{\frac{1}{p(x',0)}}}{\lambda} \right|^{p(x',0)} dx' \\
&= \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{\infty} \sum_{\nu \in \mathbb{N}_0} 2^{\nu p(x', x_n) s(x', x_n)} \left| \frac{\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x', x_n)}{\lambda} \right|^{p(x', x_n)} dx_n \right) dx' \\
&= \int_{\mathbb{R}^n} \left( \sum_{\nu \in \mathbb{N}_0} 2^{\nu p(x) s(x)} \left| \frac{\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x)}{\lambda} \right|^{p(x)} \right) dx \\
&= \int_{\mathbb{R}^n} \left\{ \left( \sum_{\nu \in \mathbb{N}_0} \left| \frac{2^{\nu s(x)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x)}{\lambda} \right|^{p(x)} \right)^{\frac{1}{p(x)}} \right\}^{p(x)} dx. \tag{63}
\end{aligned}$$

For any  $x_n \in \mathbb{R}$ , positive integers  $\nu$  satisfying  $x_n \in [2^{-\nu}, 2^{-\nu+1})$  are at most three. This implies that, by  $\tilde{Q}_{\nu, m'} = Q_{\nu, m'} \times [2^{-\nu}, 2^{-\nu+1})$ ,

$$\left( \sum_{\nu \in \mathbb{N}_0} \left| \frac{2^{\nu s(x)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x)}{\lambda} \right|^{p(x)} \right)^{\frac{1}{p(x)}}$$

consists of at most three non-zero members for any  $x \in \mathbb{R}^n$ . Hence we have

$$\begin{aligned}
& \left( \sum_{\nu \in \mathbb{N}_0} \left| \frac{2^{\nu s(x)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x)}{\lambda} \right|^{p(x)} \right)^{\frac{1}{p(x)}} \\
& \sim \left( \sum_{\nu \in \mathbb{N}_0} \left| \frac{2^{\nu s(x)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x)}{\lambda} \right|^{q(x)} \right)^{\frac{1}{q(x)}}.
\end{aligned}$$

By (63) and the equation as above, we see that

$$\begin{aligned}
& \int_{\mathbb{R}^{n-1}} \left| \frac{\left\{ \sum_{\nu \in \mathbb{N}_0} \left( 2^{\nu(s(x',0) - \frac{1}{p(x',0)})} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\nu, m'} \right)^{p(x',0)} \right\}^{\frac{1}{p(x',0)}}}{\lambda} \right|^{p(x',0)} dx' \\
&= \int_{\mathbb{R}^n} \left\{ \left( \sum_{\nu \in \mathbb{N}_0} \left| \frac{2^{\nu s(x)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x)}{\lambda} \right|^{p(x)} \right)^{\frac{1}{p(x)}} \right\}^{p(x)} dx \\
&\sim \int_{\mathbb{R}^n} \left\{ \left( \sum_{\nu \in \mathbb{N}_0} \left| \frac{2^{\nu s(x)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}(x)}{\lambda} \right|^{q(x)} \right)^{\frac{1}{q(x)}} \right\}^{p(x)} dx \tag{64}
\end{aligned}$$

holds for any  $\lambda > 0$ . This implies that (60) holds.



Finally, we prove

$$\begin{aligned} & \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m', 0)} \chi_{\tilde{Q}_{\nu, m'}} \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^q(\cdot))} \\ & \sim \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m', 0)} \chi_{\nu, m'} \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^q(\cdot))}. \end{aligned} \quad (65)$$

Without loss of generality, we can assume  $\left\| \{\delta_{m_n, 0} \lambda_{\nu, m}\}_{(\nu, m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \right\|_{f_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)} = 1$  and assume  $\lambda_{\nu, m} = 0$  when  $m_n \neq 0$ . Let  $\alpha := \frac{\min(p^-, q^-, 1)}{2}$  and let

$$\lambda' := \{\lambda_{\nu, (m', 0)}\}_{(\nu, m') \in \mathbb{N}_0 \times \mathbb{Z}^{n-1}}. \quad (66)$$

Let  $\tilde{Q}_{\nu, m'} = Q_{\nu, m'} \times [2^{-\nu}, 2^{-\nu+1})$  for  $\nu \in \mathbb{N}_0$  and  $m' \in \mathbb{Z}^{n-1}$ . If  $x \in \tilde{Q}_{\nu, m'}$  and  $y \in Q_{\nu, (m', 0)}$ , then we that  $|x - y| \leq 2\sqrt{n}2^{-\nu}$ . Therefore, we obtain

$$1 \leq \left( \frac{2\sqrt{n} + 1}{1 + 2^\nu |x - y|} \right)^M$$

for any  $M > 0$ . Let  $M > 2 \max(C_{\log}(s), n)$ . Hence, we see that

$$\begin{aligned} |\lambda_{\nu, m'}|^\alpha \chi_{\tilde{Q}_{\nu, m'}}(x) & \leq \frac{\chi_{\tilde{Q}_{\nu, m'}}(x)}{|Q_{\nu, (m', 0)}|} \int_{Q_{\nu, (m', 0)}} |\lambda_{\nu, m'}|^\alpha \chi_{\nu, (m', 0)}(y) dy \\ & \lesssim \int_{\mathbb{R}^n} \frac{2^{\nu n}}{(1 + 2^\nu |x - y|)^M} (|\lambda_{\nu, m'}|^\alpha \chi_{\nu, (m', 0)}(y)) dy \\ & \lesssim \int_{\mathbb{R}^n} \frac{2^{\nu n}}{(1 + 2^\nu |x - y|)^M} \left( \sum_{m' \in \mathbb{Z}^{n-1}} |\lambda_{\nu, m'}|^\alpha \chi_{\nu, (m', 0)}(y) \right) dy \\ & = \left( \eta_{\nu, M} * \left( \sum_{m' \in \mathbb{Z}^{n-1}} |\lambda_{\nu, m'}| \chi_{\nu, (m', 0)}(\cdot) \right)^\alpha \right) (x). \end{aligned}$$

By Lemma 2.8, we obtain

$$\left| 2^{\nu s(x)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, m'} \chi_{\tilde{Q}_{\nu, m'}}(x) \right| \lesssim \left( \eta_{\nu, \frac{M}{2}} * \left( 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} |\lambda_{\nu, m'}| \chi_{\nu, (m', 0)}(\cdot) \right)^\alpha \right)^{\frac{1}{\alpha}} (x). \quad (67)$$

By (67) and Theorem 2.5, we see that

$$\begin{aligned} & \int_{\mathbb{R}^n} \left( \sum_{\nu \in \mathbb{N}_0} \left| 2^{\nu s(x)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m', 0)} \chi_{\tilde{Q}_{\nu, m'}}(x) \right|^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \\ & \lesssim_{p, q} \int_{\mathbb{R}^n} \left( \sum_{\nu \in \mathbb{N}_0} \left| \left( \eta_{\nu, \frac{M}{2}} * \left( 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} |\lambda_{\nu, m'}| \chi_{\nu, (m', 0)}(\cdot) \right)^\alpha \right)^{\frac{1}{\alpha}} (x) \right|^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \\ & \lesssim 1. \end{aligned}$$

This implies that

$$\begin{aligned} & \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m', 0)} \chi_{\tilde{Q}_{\nu, m'}} \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^q(\cdot))} \\ & \lesssim \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m', 0)} \chi_{\nu, m'} \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^q(\cdot))}. \end{aligned} \quad (68)$$

On the other hand, without loss of generality, we can assume that

$$\left\| \left\{ 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m', 0)} \chi_{\nu, m'} \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^q(\cdot))} = 1.$$

By using same argument, we have

$$\begin{aligned} |\lambda_{\nu, m'}|^\alpha \chi_{\nu, (m', 0)}(x) & \leq \frac{\chi_{\nu, (m', 0)}(x)}{|\tilde{Q}_{\nu, m'}|} \int_{\tilde{Q}_{\nu, m'}} |\lambda_{\nu, m'}|^\alpha \chi_{\tilde{Q}_{\nu, m'}}(y) dy \\ & \lesssim \int_{\mathbb{R}^n} \frac{2^{\nu n}}{(1 + 2^\nu |x - y|)^M} (|\lambda_{\nu, m'}|^\alpha \chi_{\tilde{Q}_{\nu, m'}}(y)) dy \\ & \lesssim \int_{\mathbb{R}^n} \frac{2^{\nu n}}{(1 + 2^\nu |x - y|)^M} \left( \sum_{m' \in \mathbb{Z}^{n-1}} |\lambda_{\nu, m'}|^\alpha \chi_{\tilde{Q}_{\nu, m'}}(y) \right) dy \\ & = \left( \eta_{\nu, M} * \left( \sum_{m' \in \mathbb{Z}^{n-1}} |\lambda_{\nu, m'}| \chi_{\tilde{Q}_{\nu, m'}}(\cdot) \right)^\alpha \right)(x). \end{aligned}$$

Hence we see that

$$\left| 2^{\nu s(x)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, m'} \chi_{\nu, (m', 0)}(x) \right| \lesssim \left( \eta_{\nu, \frac{M}{2}} * \left( 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} |\lambda_{\nu, m'}| \chi_{\tilde{Q}_{\nu, m'}}(\cdot) \right)^\alpha \right)^{\frac{1}{\alpha}}(x) \quad (69)$$

By (69) and Theorem 2.5, we see that

$$\begin{aligned} & \int_{\mathbb{R}^n} \left( \sum_{\nu \in \mathbb{N}_0} \left| 2^{\nu s(x)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m', 0)} \chi_{\nu, (m', 0)}(x) \right|^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \\ & \lesssim_{p, q} \int_{\mathbb{R}^n} \left( \sum_{\nu \in \mathbb{N}_0} \left| \left( \eta_{\nu, \frac{M}{2}} * \left( 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} |\lambda_{\nu, m'}| \chi_{\tilde{Q}_{\nu, m'}}(\cdot) \right)^\alpha \right)^{\frac{1}{\alpha}}(x) \right|^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \\ & \lesssim 1. \end{aligned}$$

This implies that

$$\begin{aligned} & \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m', 0)} \chi_{\tilde{Q}_{\nu, m'}} \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^q(\cdot))} \\ & \gtrsim \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m', 0)} \chi_{\nu, m'} \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^q(\cdot))}. \end{aligned} \quad (70)$$

By (68) and (70), we have (65). Therefore, Lemma 5.6 holds by (60) and (65).  $\square$

**Lemma 5.7.** *Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ . For double-index complex-valued sequence  $\lambda = \{\lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n}$ , we have*

$$\left\| \{\lambda_{\nu,(m',0)}\}_{(\nu,m') \in \mathbb{N}_0 \times \mathbb{Z}^{n-1}} \right\|_{b_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1})} \sim \left\| \{\delta_{m_n,0} \lambda_{\nu,m}\}_{(\nu,m) \in \mathbb{N}_0 \times \mathbb{Z}^n} \right\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)}.$$

*Proof.* By Corollary 5.5, it suffices to consider the case  $p(\cdot)$ ,  $q(\cdot)$  and  $s(\cdot)$  independent of the  $n$ -th coordinate for  $|x_n| \leq 2$ . Let  $\tilde{Q}_{\nu,m'} = Q_{\nu,m'} \times [2^{-\nu}, 2^{-\nu+1})$  for  $\nu \in \mathbb{N}_0$  and  $m' \in \mathbb{Z}^{n-1}$ . Firstly, we prove

$$\begin{aligned} & \left\| \left\{ 2^{\nu(\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)})} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu,(m',0)} \chi_{\nu,m'} \right\}_{\nu=0}^{\infty} \right\|_{\ell^{\tilde{q}(\cdot)}(L^{\tilde{p}(\cdot)})} \\ & \sim \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu,(m',0)} \chi_{\tilde{Q}_{\nu,m'}} \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}. \end{aligned} \quad (71)$$

Let  $\lambda > 0$  and  $\mu > 0$ . Then we recall that

$$\begin{aligned} & \left\| \left\{ 2^{\nu(\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)})} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu,(m',0)} \chi_{\nu,m'} \right\}_{\nu=0}^{\infty} \right\|_{\ell^{\tilde{q}(\cdot)}(L^{\tilde{p}(\cdot)})} \\ & = \inf \left\{ \lambda > 0 : \sum_{\nu=0}^{\infty} \left\| \left( \frac{2^{\nu(\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)})} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu,(m',0)} \chi_{\nu,m'}}{\lambda} \right)^{\tilde{q}(\cdot)} \right\|_{L^{\frac{\tilde{p}(\cdot)}{\tilde{q}(\cdot)}}} \right\} \end{aligned}$$

and

$$\begin{aligned} & \left\| \left( \frac{2^{\nu(\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)})} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu,(m',0)} \chi_{\nu,m'}}{\lambda} \right)^{\tilde{q}(\cdot)} \right\|_{L^{\frac{\tilde{p}(\cdot)}{\tilde{q}(\cdot)}}} \\ & = \inf \left\{ \mu > 0 : \int_{\mathbb{R}^{n-1}} \left\{ \frac{\left( 2^{\nu(s(x',0) - \frac{1}{p(x',0)})} \sum_{m' \in \mathbb{Z}^{n-1}} \frac{\lambda_{\nu,(m',0)}}{\lambda} \chi_{\nu,m'} \right)^{q(x',0)}}{\mu} \right\}^{\frac{p(x',0)}{q(x',0)}} dx' \right\} \end{aligned}$$

for each  $\nu \in \mathbb{N}_0$ . By  $2^{-\nu} = \int_{2^{-\nu}}^{2^{-\nu+1}} \chi_{[2^{-\nu}, 2^{-\nu+1})} dx$ , we see that

$$\begin{aligned}
& \int_{\mathbb{R}^{n-1}} \left\{ \frac{\left( 2^{\nu(s(x',0) - \frac{1}{p(x',0)})} \sum_{m' \in \mathbb{Z}^{n-1}} \frac{\lambda_{\nu, (m',0)}}{\lambda} \chi_{\nu, m'} \right)^{q(x',0)}}{\mu} \right\}^{\frac{p(x',0)}{q(x',0)}} dx' \\
&= \int_{\mathbb{R}^{n-1}} \frac{\left( 2^{\nu(s(x',0) - \frac{1}{p(x',0)})} \sum_{m' \in \mathbb{Z}^{n-1}} \frac{\lambda_{\nu, (m',0)}}{\lambda} \chi_{\nu, m'} \right)^{p(x',0)}}{\mu^{\frac{p(x',0)}{q(x',0)}}} dx' \\
&= \int_{\mathbb{R}^{n-1}} \frac{2^{\nu(s(x',0)q(x',0)-1)} \left( \sum_{m' \in \mathbb{Z}^{n-1}} \frac{\lambda_{\nu, (m',0)}}{\lambda} \chi_{\nu, m'} \right)^{p(x',0)}}{\mu^{\frac{p(x',0)}{q(x',0)}}} dx' \\
&= \int_{\mathbb{R}^{n-1}} \left\{ \int_{2^{-\nu}}^{2^{-\nu+1}} \frac{2^{\nu s(x',0)q(x',0)} \left( \sum_{m' \in \mathbb{Z}^{n-1}} \frac{\lambda_{\nu, (m',0)}}{\lambda} \chi_{\nu, m'} \right)^{p(x',0)} \chi_{[2^{-\nu}, 2^{-\nu+1})}(x_n)}{\mu^{\frac{p(x',0)}{q(x',0)}}} dx_n \right\} dx' \\
&= \int_{\mathbb{R}^{n-1}} \left\{ \int_{2^{-\nu}}^{2^{-\nu+1}} \frac{2^{\nu s(x',x_n)q(x',x_n)} \left( \sum_{m' \in \mathbb{Z}^{n-1}} \frac{\lambda_{\nu, (m',0)}}{\lambda} \chi_{\tilde{Q}_{\nu, m'}} \right)^{p(x',x_n)}}{\mu^{\frac{p(x',x_n)}{q(x',x_n)}}} dx_n \right\} dx' \\
&= \int_{\mathbb{R}^n} \frac{2^{\nu s(x)q(x)} \left( \sum_{m' \in \mathbb{Z}^{n-1}} \frac{\lambda_{\nu, (m',0)}}{\lambda} \chi_{\tilde{Q}_{\nu, m'}} \right)^{p(x)}}{\mu^{\frac{p(x)}{q(x)}}} dx \\
&= \int_{\mathbb{R}^n} \left( \frac{2^{\nu s(x)q(x)} \left( \sum_{m' \in \mathbb{Z}^{n-1}} \frac{\lambda_{\nu, (m',0)}}{\lambda} \chi_{\tilde{Q}_{\nu, m'}} \right)^{q(x)}}{\mu} \right)^{\frac{p(x)}{q(x)}} dx
\end{aligned}$$

holds for any  $\nu \in \mathbb{N}_0$ . This implies that

$$\left\| \left( \frac{2^{\nu(s(\cdot) - \frac{1}{p(\cdot)})} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\nu, m'}}{\lambda} \right)^{\tilde{q}(\cdot)} \right\|_{L^{\frac{\tilde{p}(\cdot)}{\tilde{q}(\cdot)}}} = \left\| \left( \frac{2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}}}{\lambda} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}$$

holds for any  $\nu \in \mathbb{N}_0$ . Therefore, we obtain (71).

Secondly, we prove

$$\begin{aligned}
& \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\tilde{Q}_{\nu, m'}} \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\
& \sim \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, (m',0)} \chi_{\nu, m'} \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}. \tag{72}
\end{aligned}$$

By using the same argument of the proof of Lemma 5.6, we have (67) and (69). Hence we obtain (72).

Therefore, Lemma 5.7 holds by (71) and (72).

□

Now, we can prove Theorem 5.1.

*Proof of Theorem 5.1.* Firstly, we prove that  $\text{Tr}_{\mathbb{R}^n}$  is well-defined. We apply Theorem 4.17 as  $f \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ , we have

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta (\beta \text{qu})_{\nu, m}.$$

Here we can take  $\psi$  in Definition 4.16 such that  $\psi(x) = \mu(x_1)\mu(x_2)\cdots\mu(x_n)$ , where  $\mu$  is a 1-dimensional smooth function such that  $\text{supp}(\mu) \subset (-1, 1)$ . By using the quarkonial decomposition, we extend  $\text{Tr}_{\mathbb{R}^n}$  so that

$$\text{Tr}_{\mathbb{R}^n} f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta (\beta \text{qu})_{\nu, m}(\cdot, 0).$$

By the support of  $\mu$  and the definition of quarks,  $\beta_n$  and  $m_n$  are 0. Therefore, we have

$$\text{Tr}_{\mathbb{R}^n} f = \sum_{\beta \in \mathbb{N}_0^n, \beta_n=0} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n, m_n=0} \lambda_{\nu, m}^\beta (\beta \text{qu})_{\nu, m}(\cdot, 0). \quad (73)$$

Let  $\lambda^{\beta'} = \{\lambda_{\nu, (m', 0)}^{(\beta', 0)}\}$ . Then we see that extended  $\text{Tr}_{\mathbb{R}^n}$  is well-defined because that the convergence of (73) is uniformly if  $f \in \mathcal{S}(\mathbb{R}^n)$  and that

$$\|\lambda^{\beta'}\|_{f_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1})} \lesssim \|\lambda^\beta\|_{f_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)}$$

and

$$\|\lambda^{\beta'}\|_{b_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1})} \lesssim \|\lambda^\beta\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)}$$

hold by Lemma 5.6.

Since we assume (40), we can take  $\epsilon > 0$  such that  $0 < \epsilon < \rho - r$ . For any  $\nu \in \mathbb{N}_0$  and  $m \in \mathbb{Z}^n$ , there exists  $d > 0$  such that  $\text{supp}(\beta \text{qu})_{\nu, m} \subset dQ_{\nu, m}$ . The condition (54) implies that smooth atoms in  $F_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1})$  are not required to satisfy any moment conditions and that smooth atoms in  $B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1})$  are not required to satisfy any moment conditions. Hence we can regard  $2^{-(r+\epsilon)|\beta|}(\beta \text{qu})_{\nu, m}$  as smooth atoms in  $F_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1})$  and  $B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1})$ .

Let  $\alpha = \min(p^-, q^-, 1)$ . Then we see that

$$\begin{aligned} \|\text{Tr}_{\mathbb{R}^n} f\|_{F_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}}^\alpha &\leq \sum_{\beta \in \mathbb{N}_0^n, \beta_n=0} \left\| \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n, m_n=0} \lambda_{\nu, m}^\beta (\beta \text{qu})_{\nu, m}(\cdot, 0) \right\|_{F_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}}^\alpha \\ &\leq \sum_{\beta'} \|\lambda^{\beta'}\|_{f_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}}^\alpha \\ &\lesssim \sum_{\beta \in \mathbb{N}_0} \|\lambda^\beta\|_{f_{p(\cdot), q(\cdot)}^{s(\cdot)}}^\alpha \\ &\lesssim \sum_{\beta \in \mathbb{N}_0} 2^{-\rho\alpha|\beta|} \|\lambda\|_{f_{p(\cdot), q(\cdot), \rho}^{s(\cdot)}}^\alpha \\ &\lesssim \sum_{\beta \in \mathbb{N}_0} 2^{-\rho\alpha|\beta|} \|\lambda\|_{f_{p(\cdot), q(\cdot), \rho}^{s(\cdot)}}^\alpha \\ &\lesssim \|f\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}}^\alpha \end{aligned}$$

by Lemma 2.1. Therefore, we have  $\|\text{Tr}_{\mathbb{R}^n} f\|_{F_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}} \lesssim \|f\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}}$ . By using same calculation,

we also have  $\|\text{Tr}_{\mathbb{R}^n} f\|_{B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}} \lesssim \|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot)}}$ . Surjective follows from second assertion with  $k = 0$ .

Finally, we prove the second assertion for Triebel–Lizorkin spaces. We fix  $j = 0, 1, 2, \dots, k$ . Let  $g_j \in F_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot)-j-\frac{1}{p(\cdot)}}(\mathbb{R}^{n-1})$ . We can write

$$g_j = \sum_{\beta' \in \mathbb{N}_0^{n-1}} \sum_{\nu \in \mathbb{N}_0} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, m'}^{\beta'} (\beta' \text{qu})_{\nu, m'} \quad (74)$$

by Theorem 4.17. Let  $L \gg 1$  and

$$v_j = \sum_{\beta' \in \mathbb{N}_0^{n-1}} \sum_{\nu \in \mathbb{N}_0} \sum_{m' \in \mathbb{Z}^{n-1}} \frac{\lambda_{\nu, m'}^{\beta'}}{(2L+j)! 2^{\nu(2L+j)}} ((\beta', 2L+j) \text{qu})_{\nu, (m', 0)}$$

Then we see that  $v_j \in F_{p(\cdot), q(\cdot)}^{s(\cdot)+2L}(\mathbb{R}^n)$  by Theorem 4.17 and Lemma 5.6. We define  $h_j = \partial_{x_n}^{2L} v_j \in F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  such that

$$h_j := \sum_{\beta' \in \mathbb{N}_0^{n-1}} \sum_{\nu \in \mathbb{N}_0} \sum_{m' \in \mathbb{Z}^{n-1}} \frac{\lambda_{\nu, m'}^{\beta'}}{(2L+j)!} \partial_{x_n}^{2L} [x_n^{2L+j} ((\beta', 0) \text{qu})_{\nu, (m', 0)}]. \quad (75)$$

Then we see that the right hand side of

$$\partial_{x_n}^l h_j = \frac{1}{(2L+j)!} \sum_{\beta' \in \mathbb{N}_0^{n-1}} \sum_{\nu \in \mathbb{N}_0} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, m'}^{\beta'} \partial_{x_n}^{2L+l} [x_n^{2L+j} ((\beta', 0) \text{qu})_{\nu, (m', 0)}] \quad (76)$$

converges in the sense of  $F_{p(\cdot), q(\cdot)}^{s(\cdot)-l}(\mathbb{R}^n)$  for any  $l = 0, 1, \dots, k$ . Now we consider the

$$\text{Tr}_{\mathbb{R}^n} [\partial_{x_n}^l h_j] = \frac{1}{(2L+j)!} \text{Tr}_{\mathbb{R}^n} \left[ \sum_{\beta' \in \mathbb{N}_0^{n-1}} \sum_{\nu \in \mathbb{N}_0} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, m'}^{\beta'} \partial_{x_n}^{2L+l} [x_n^{2L+j} ((\beta', 0) \text{qu})_{\nu, (m', 0)}] \right]. \quad (77)$$

We can change the order  $\text{Tr}_{\mathbb{R}^n}$  and summation in (77) because  $\text{Tr}_{\mathbb{R}^n}$  is continuous on  $F_{p(\cdot), q(\cdot)}^{s(\cdot)-l}(\mathbb{R}^n)$ . Hence we have

$$\text{Tr}_{\mathbb{R}^n} [\partial_{x_n}^l h_j] = \frac{1}{(2L+j)!} \sum_{\beta' \in \mathbb{N}_0^{n-1}} \sum_{\nu \in \mathbb{N}_0} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, m'}^{\beta'} \text{Tr}_{\mathbb{R}^n} \left[ \partial_{x_n}^{2L+l} [x_n^{2L+j} ((\beta', 0) \text{qu})_{\nu, (m', 0)}] \right]. \quad (78)$$

Let

$$\delta_{j,l} = \begin{cases} 1 & j = l, \\ 0 & j \neq l. \end{cases}$$

Recall that the definition of  $\psi(x)$  and quarks, we have  $\sum_{m \in \mathbb{Z}^n} \mu(x_n - m) = 1$ . Furthermore, by the support of  $\mu$ , it is easy to see that  $\mu^{(k)}(0) = 0$  for any  $k \in \mathbb{N}$ . Hence we have

$$\text{Tr}_{\mathbb{R}^n} \left[ \partial_{x_n}^{2L+l} [x_n^{2L+j} ((\beta', 0) \text{qu})_{\nu, m'}(x')] \right] = \delta_{j,l} (2L+j)! ((\beta', 0) \text{qu})_{\nu, m'}(x'). \quad (79)$$

Therefore, we obtain

$$\begin{aligned} \text{Tr}_{\mathbb{R}^n} [\partial_{x_n}^l h_j] &= \frac{1}{(2L+j)!} \sum_{\beta' \in \mathbb{N}_0^{n-1}} \sum_{\nu \in \mathbb{N}_0} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu, m'}^{\beta'} \text{Tr}_{\mathbb{R}^n} \partial_{x_n}^{2L+l} [x_n^{2L+j} ((\beta', 0) \text{qu})_{\nu, (m', 0)}] \\ &= \delta_{j,l} g_j \end{aligned}$$

by (78) and (79). This implies that  $f = \sum_{j=0}^k h_j$  satisfies the second assertion of Triebel–Lizorkin case. We can also prove the Besov case by using similar argument as above.  $\square$

## 6. TRACE THEOREM FOR UPPER HALF SPACE $\mathbb{R}_+^n$

We will extend Theorem 5.1 to Besov spaces and Triebel–Lizorkin spaces with variable exponents on  $\mathbb{R}_+^n = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$ . To do this, we define spaces  $B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$  and  $F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$ .

**Definition 6.1.** Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ .

Besov spaces with variable exponents on upper half plane  $B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$  is the collection of  $f \in \mathcal{D}'(\mathbb{R}_+^n)$  such that there exists a  $g \in B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  satisfying  $f = g|_{\mathbb{R}_+^n}$ , where  $g|_{\mathbb{R}_+^n}$  denotes the restriction of  $g \in \mathcal{S}'(\mathbb{R}^n)$  to  $\mathbb{R}_+^n$  in the sense of  $\mathcal{D}'(\mathbb{R}_+^n)$ . The space  $B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$  becomes a normed space equipped with the norm

$$\|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)} := \inf \left\{ \|g\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)} : g \in B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n), f = g|_{\mathbb{R}_+^n} \right\}.$$

Triebel–Lizorkin spaces with variable exponents on upper half plane  $F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$  is the collection of  $f \in \mathcal{D}'(\mathbb{R}_+^n)$  such that there exists a  $g \in F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  satisfying  $f = g|_{\mathbb{R}_+^n}$ . The space  $F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$  becomes a normed space equipped with the norm

$$\|f\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)} := \inf \left\{ \|g\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)} : g \in F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n), f = g|_{\mathbb{R}_+^n} \right\}.$$

Then we have following theorem as well as  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  cases.

**Theorem 6.2.** Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Then

(1)  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$  is a quasi-Banach space.

(2) For any  $f, g \in F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$ , we have

$$\|f + g\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)}^{\min(p^-, q^-, 1)} \leq \|f\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)}^{\min(p^-, q^-, 1)} + \|g\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)}^{\min(p^-, q^-, 1)}.$$

(3) Let

$$\alpha = \min(q^-, 1) \min\left(1, \left(\frac{p}{q}\right)^-\right).$$

Then, for any  $f, g \in B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$ , we have

$$\|f + g\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)}^\alpha \leq \|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)}^\alpha + \|g\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)}^\alpha.$$

**6.1. Extension of Franke and Runst's lift operator.** In this subsection, we extend the lifting operator introduced by Franke and Runst [9]. We construct a collection of operator  $\{J_\sigma\}_{\sigma \in \mathbb{R}}$  such that following three conditions.

- (1)  $J_\sigma$  is an isomorphism between  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  and  $A_{p(\cdot), q(\cdot)}^{s(\cdot) - \sigma}(\mathbb{R}^n)$ .
- (2)  $J_\sigma$  is an inverse map of  $J_{-\sigma}$ .
- (3) Let  $f \in \mathcal{S}'(\mathbb{R}^n)$ . If

$$\text{supp } f \subset \mathbb{R}^{n-1} \times (-\infty, 0],$$

then

$$\text{supp } J_\sigma f \subset \mathbb{R}^{n-1} \times (-\infty, 0].$$

Let  $\eta \in \mathcal{S}(\mathbb{R}^n)$  be a positive function which satisfy  $\text{supp } \eta \subseteq (-2, -1)$  and  $\int_{\mathbb{R}^n} \eta(x) dx = 2$ . For any  $0 < \epsilon \ll 1$ , we define a holomorphic function  $\psi_\epsilon$  on  $\mathbb{C}$  such that

$$\psi_\epsilon(x) := \int_{-\infty}^0 \eta(t) e^{-i\epsilon t x} dt - iz.$$

Let  $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ ,  $\overline{\mathbb{H}} = \{z \in \mathbb{C} : \text{Im}(z) \geq 0\}$  and  $\Omega = \{z \in \mathbb{C} : |z| > 4, \text{Re}(z) > 0\}$ . If  $z \in \mathbb{C}$  satisfy  $|z| > 4$  and  $\text{Im}(z) > 0$ , then  $-iz \in \Omega$ . Hence we see

$$\text{dist}(\psi_\epsilon(z), \Omega) \leq |\psi_\epsilon(z) + iz| = \left| \int_{-\infty}^0 \eta(t) e^{-i\epsilon t z} dt \right| < 2. \quad (80)$$

If  $z \in \mathbb{C}$  satisfy  $|z| \leq 4$  and  $\text{Im}(z) \geq 0$ , then we have  $\text{Re}(\psi_0(z)) = 2 + \text{Im}(z)$ . Therefore, for any  $0 < \epsilon \ll 1$ , we obtain

$$\text{Re}(\psi_\epsilon(z)) = \int_{-\infty}^0 \eta(t) e^{\epsilon \text{Im}(z)} \cos(\epsilon t \text{Re}(z)) dt + \text{Im}(z) \geq \frac{3}{2}.$$

If  $\epsilon > 0$  is a sufficiency small real number, we see that  $\psi_\epsilon$  is a mapping from  $\overline{\mathbb{H}}$  to

$$\Omega_0 := \{z \in \mathbb{C} : \text{Re}(z) > 1\} \cup \{z \in \mathbb{C} : |\text{Im}(z)| > 1\}. \quad (81)$$

We fix such a sufficiency small real number  $\epsilon > 0$ . We select a branch-cut of  $\log$  on simply-connected region  $\mathbb{C} \setminus (-\infty, 0]$  such that  $\log 1 = 0$ . Then we define  $a^z = \exp(a \log z)$  for  $z \in \mathbb{C} \setminus (-\infty, 0]$ . Furthermore, for any  $\sigma \in \mathbb{R}$ , we define  $\varphi^{(\sigma)} : \mathbb{R}^{n-1} \times \overline{\mathbb{H}} \rightarrow \mathbb{C}$  such that

$$\varphi^{(\sigma)}(x', z_n) := \left( \langle x' \rangle \psi_\epsilon \left( \frac{z_n}{\langle x' \rangle} \right) \right)^\sigma, \quad z \in \overline{\mathbb{H}}. \quad (82)$$

We put  $\varphi := \varphi^{(1)}$ . Then we have following lemma.

**Lemma 6.3.** *For any  $\alpha \in \mathbb{N}_0^n$ ,*

$$|\partial^\alpha \varphi(x', z_n)| \lesssim_\alpha (\langle x' \rangle + |z_n|)^{1-|\alpha|}, \quad (x', z_n) \in \mathbb{R}^{n-1} \times \overline{\mathbb{H}} \quad (83)$$

*holds. Furthermore, for any  $(x', z_n) \in \mathbb{R}^{n-1} \times \overline{\mathbb{H}}$ , we have*

$$\langle x' \rangle + |z_n| \sim |\varphi(x', z_n)|. \quad (84)$$

*Proof.* Firstly, we will prove (84). If  $|z_n| > 4\langle x' \rangle$ , then we obtain

$$\left| \psi_\epsilon \left( \frac{|z_n|}{\langle x' \rangle} \right) + i \frac{z_n}{\langle x' \rangle} \right| < 2 < \frac{|z_n|}{2\langle x' \rangle}$$

by (80). This implies

$$\frac{|z_n|}{2\langle x' \rangle} < \left| \psi_\epsilon \left( \frac{|z_n|}{\langle x' \rangle} \right) \right| < \frac{3|z_n|}{2\langle x' \rangle}.$$

If  $|z_n| \leq 4\langle x' \rangle$ , then we see that

$$\left| \psi_\epsilon \left( \frac{|z_n|}{\langle x' \rangle} \right) \right| \leq \int_{-\infty}^0 \eta(t) dt + \frac{|z_n|}{\langle x' \rangle} \leq 2 + 4 = 6.$$

Hence (84) holds.

Finally, we will prove (83). By the definition of  $\varphi(x', z_n)$ , we have

$$\varphi(x', z_n) = \langle x' \rangle \int_{-\infty}^0 \eta(t) \exp \left( -it\epsilon \frac{z_n}{\langle x' \rangle} \right) dt - iz_n. \quad (85)$$



Here, it is easy to see that the second term of the right hand side of (85) satisfies (83). Hence, we estimate the first term of the right hand side of (85). By Libniz's formula, we see that

$$\begin{aligned} \left| \partial^\alpha \left( \langle x' \rangle \int_{-\infty}^0 \eta(t) \exp \left( -it\epsilon \frac{z_n}{\langle x' \rangle} \right) dt \right) \right| &\lesssim \sum_{\gamma \leq \alpha} \left| \partial^{\alpha-\gamma} \langle x' \rangle \partial^\gamma \left( \int_{-\infty}^0 \eta(t) \exp \left( -it\epsilon \frac{z_n}{\langle x' \rangle} \right) dt \right) \right| \\ &\lesssim \sum_{\gamma \leq \alpha} \langle x' \rangle^{1-|\alpha|+|\gamma|} \left| \partial^\gamma \left( \int_{-\infty}^0 \eta(t) \exp \left( -it\epsilon \frac{z_n}{\langle x' \rangle} \right) dt \right) \right|. \end{aligned}$$

Since

$$\int_{-\infty}^0 \eta(t) \exp \left( -it\epsilon \frac{z_n}{\langle x' \rangle} \right) dt = \sqrt{2\pi} \mathcal{F} \left[ \eta \exp \left( t\epsilon \frac{\text{Im}(z_n)}{\langle x' \rangle} \right) \right] \left( t\epsilon \frac{\text{Re}(z_n)}{\langle x' \rangle} \right),$$

we obtain

$$\left| \partial^\alpha \left( \langle x' \rangle \int_{-\infty}^0 \eta(t) \exp \left( -it\epsilon \frac{z_n}{\langle x' \rangle} \right) dt \right) \right| \lesssim \langle x' \rangle^{1-|\alpha|} \left( 1 + \frac{|z_n|}{\langle x' \rangle} \right)^{1-|\alpha|} = (\langle x' \rangle + |z_n|)^{1-|\alpha|}.$$

□

We also use same symbol  $\varphi^{(\sigma)}$  for  $\varphi^{(\sigma)}|_{\mathbb{R}^{n-1} \times \mathbb{R}}$ . Then by Theorem 2.11 and Lemma 6.3, we have following Proposition.

**Proposition 6.4.** *Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Then, for any  $\sigma \in \mathbb{R}$ , we have following properties.*

- (1)  $J_\sigma := \varphi^{(\sigma)}(D)$  is a linear isomorphism between  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  and  $A_{p(\cdot), q(\cdot)}^{s(\cdot)-\sigma}(\mathbb{R}^n)$ .
- (2)  $J_{-\sigma}$  is the inverse operator of  $J_\sigma$ .
- (3) For any  $f \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ , we have

$$\|J_\sigma f\|_{A_{p(\cdot), q(\cdot)}^{s(\cdot)-\sigma}(\mathbb{R}^n)} \sim \|f\|_{A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)}.$$

To consider the support of  $J_\sigma f$ , we use following the Paley–Wiener theorem.

**Lemma 6.5** (Paley–Wiener theorem). *Let  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  and  $\epsilon > 0$ . Then,  $\text{supp } \varphi \subset \mathbb{R}^{n-1} \times [\epsilon, \infty)$  if and only if  $\mathcal{F}^{-1}\varphi$  can be extended to a continuous function  $\Psi : \mathbb{R}^{n-1} \times \overline{\mathbb{H}} \rightarrow \mathbb{C}$  satisfying*

- (1)  $\Psi(\xi', \cdot)$  is a holomorphic function on  $\mathbb{H}$ ,
- (2) For each  $N \in \mathbb{N}$ ,

$$|\Psi(\xi', \xi_n + i\zeta_n)| \lesssim_N \langle \xi \rangle^{-N} (1 + \zeta_n)^{-N} \exp(-\epsilon \zeta_n) \quad (86)$$

holds for any  $(\xi', \xi_n + i\zeta_n) \in \mathbb{R}^{n-1} \times \overline{\mathbb{H}}$ .

By using the Paley–Wiener theorem, we obtain a result about the support of  $J_\sigma f$ .

**Proposition 6.6.** *Let  $f \in \mathcal{S}'(\mathbb{R}^n)$ . If  $\text{supp } f \subset \mathbb{R}^{n-1} \times (-\infty, 0]$ , then  $\text{supp } J_\sigma f \subset \mathbb{R}^{n-1} \times (-\infty, 0]$ .*

*Proof.* We take a test function  $\psi \in \mathcal{D}(\mathbb{R}^{n-1} \times (0, \infty))$ . Since  $\psi$  has a compact support, we see that there exists an  $\epsilon > 0$  such that  $\text{supp } \psi \subset \mathbb{R}^{n-1} \times [\epsilon, \infty)$ . Then we have

$$\langle J_\sigma f, \psi \rangle = \langle f, \mathcal{F}[\varphi^{(\sigma)} \mathcal{F}^{-1}\psi] \rangle. \quad (87)$$

We can see that  $\mathcal{F}^{-1}\psi$  satisfies (86) and that  $\varphi^{(\sigma)} \mathcal{F}^{-1}\psi$  also satisfies (86) by Lemma 6.3. Hence, we obtain  $\text{supp } \mathcal{F}[\varphi^{(\sigma)} \mathcal{F}^{-1}\psi] \subset \mathbb{R}^{n-1} \times [\epsilon, \infty)$ . Therefore, we have  $\langle J_\sigma f, \psi \rangle = 0$  because  $\text{supp } f \subset \mathbb{R}^{n-1} \times (-\infty, 0]$ . □

**Theorem 6.7.** *Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ ,  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$  and  $\sigma \in \mathbb{R}$ .*

(1) *Let  $f \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$ . Then  $J_\sigma f := (J_\sigma g)|_{\mathbb{R}_+^n}$  does not depend on  $g \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  satisfying  $f = g|_{\mathbb{R}_+^n}$ .*

(2)  *$J_\sigma$  is an isomorphism between  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$  and  $A_{p(\cdot), q(\cdot)}^{s(\cdot)-\sigma}(\mathbb{R}_+^n)$ . Furthermore,  $J_{-\sigma}$  is the inverse of  $J_\sigma$ .*

*Proof.* We will prove (1). Let  $g_1, g_2 \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  satisfy  $f = g_1|_{\mathbb{R}_+^n} = g_2|_{\mathbb{R}_+^n}$ . Then we have

$$(J_\sigma(g_1 - g_2))|_{\mathbb{R}_+^n} = 0$$

by  $(g_1 - g_2)|_{\mathbb{R}_+^n} = 0$  and Proposition 6.6. Therefore, we obtain

$$(J_\sigma g_1)|_{\mathbb{R}_+^n} = (J_\sigma g_2)|_{\mathbb{R}_+^n} \quad (88)$$

because

$$(J_\sigma g_1)|_{\mathbb{R}_+^n} - (J_\sigma g_2)|_{\mathbb{R}_+^n} = (J_\sigma(g_1 - g_2))|_{\mathbb{R}_+^n} = 0.$$

(88) means  $J_\sigma f$  does not depend on  $g \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  satisfying  $f = g|_{\mathbb{R}_+^n}$ . (2) follows from the properties of  $J_\sigma$  as an operator on  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ .  $\square$

## 6.2. An extension operator for $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$ .

**Theorem 6.8.** *Let  $N \in \mathbb{N}$ ,  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Then there exists an operator  $\text{Ext}_N$  which is so called extension operator:*

$$\text{Ext}_N : \bigcup_{\substack{p(\cdot), q(\cdot) : N^{-1} \leq p^-, q^- \\ s(\cdot) : s^+ \leq N}} A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n) \longrightarrow \bigcup_{\substack{p(\cdot), q(\cdot) : N^{-1} \leq p^-, q^- \\ s(\cdot) : s^+ \leq N}} A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n), \quad (89)$$

satisfying the following conditions.

- (1)  $\text{Ext}_N|_{A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)}$  is continuous.
- (2) For any  $f \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$ ,  $(\text{Ext}_N f)|_{\mathbb{R}_+^n} = f$ .

*Proof.* Let  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$  satisfy  $N^{-1} \leq p^-, q^-$  and  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$  satisfy  $\|s\|_\infty < N$ .

**Step 1.** Let  $M \in \mathbb{N}$  be a sufficiency large. Let  $\delta_{0,l}$  be the Kronecker delta function. We define  $\lambda_1, \lambda_2, \dots, \lambda_M \in \mathbb{R}$  uniquely such that the following simultaneous equations

$$\sum_{j=0}^M (-j)^l \lambda_j = \delta_{0,l}, \quad l = 0, 1, \dots, M-1 \quad (90)$$

hold. Since a discriminant  $D$  of this simultaneous equation is some positive constant times Vandermonde determinant  $\det\{j^i\}_{i,j=1,\dots,M}$ , we have  $D \neq 0$ . Hence  $\lambda_1, \lambda_2, \dots, \lambda_M$  can be determined uniquely.

For a function  $f : \mathbb{R}^{n-1} \times [0, \infty) \longrightarrow \mathbb{C}$ , we define

$$f^*(x) := \begin{cases} f(x) & (\text{if } x_n \geq 0), \\ \sum_{j=1}^M \lambda_j f(x', -jx_n) & (\text{if } x_n \leq 0). \end{cases} \quad (91)$$

Let  $f \in \mathcal{B}^M(\mathbb{R}^n)$  be defined a neighborhood of  $\mathbb{R}^{n-1} \times [0, \infty)$ . Then  $f^* \in \mathcal{B}(\mathbb{R}^n)$  because we defined  $f^* : \mathbb{R}^n \longrightarrow \mathbb{C}$  such that the differential coefficient of  $f^*$  at boundary coincides with  $f$ .

**Step 2.** We consider the case that  $s(\cdot)$  satisfy  $0 > \left(\frac{n}{p(\cdot)} - s(\cdot)\right)^+$ . In this case, we have  $A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n) \hookrightarrow \text{BUC}(\mathbb{R}^n)$  by Proposition 3.3.

Let  $M \in \mathbb{N}_0$  as in Step 1 be sufficiently large enough to  $M \gg (n+1)N$ . For  $f \in A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$ , there exist  $g \in A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  such that  $f = g|_{\mathbb{R}_+^n}$  and

$$\|g\|_{A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)} \leq 2\|f\|_{A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)}.$$

Let  $\rho, r$  as in Theorem 4.17. Then, by Theorem 4.17, we can express  $g$  as

$$g = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m}^\beta (\beta \text{qu})_{\nu,m}. \quad (92)$$

We have  $\|\lambda\|_{a_{p(\cdot),q(\cdot),\rho}^{s(\cdot)}} \lesssim \|g\|_{A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)}$ , where  $\lambda = \{\lambda_{\nu,m}^\beta\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n}$ . Then we define  $g^*$  so that

$$g^* := \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m}^\beta (\beta \text{qu})_{\nu,m}^*. \quad (93)$$

Let  $h \in A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  satisfy  $f = h|_{\mathbb{R}_+^n}$ . Then, by Theorem 4.17, we can also express  $h$  as

$$h = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \rho_{\nu,m}^\beta (\beta \text{qu})_{\nu,m}. \quad (94)$$

We also define  $h^*$  so that

$$h^* = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \rho_{\nu,m}^\beta (\beta \text{qu})_{\nu,m}^*. \quad (95)$$

Then, we prove  $g^* = h^*$  in the sense of  $\mathcal{S}'(\mathbb{R}^n)$ . That is, we prove that  $g^*$  depend only on  $f$ . Since  $A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n) \hookrightarrow \text{BUC}(\mathbb{R}^n)$  and  $0 > \left(\frac{n}{p(\cdot)} - s(\cdot)\right)^+$ ,  $g^*$  and  $h^*$  are uniformly continuous functions. Hence, it is sufficient to prove that  $g^*(x) = h^*(x)$  for any  $x \in \mathbb{R}^n$ . Since  $g$  and  $h$  are continuous functions and  $g|_{\mathbb{R}_+^n} = h|_{\mathbb{R}_+^n}$  in the sense of  $\mathcal{D}'(\mathbb{R}^n)$ ,  $g(x) = h(x)$  for any  $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ , where  $x_n > 0$ . By the continuity of  $g(x)$  and  $h(x)$ , we have  $g(x) = h(x)$  for any  $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ , where  $x_n \geq 0$ . This implies that  $g^*(x) = h^*(x)$  for any  $x \in \mathbb{R}_+^n$ . By the definition of  $g^*(x)$  and  $h^*(x)$ , we have  $g^*(x) = h^*(x)$  for any  $x \in \mathbb{R}^n$ .

Since  $g^*$  depend only on  $f$ , we can consider

$$\text{Ext}_N f := \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m}^\beta (\beta \text{qu})_{\nu,m}^* \quad (96)$$

and

$$\text{Ext}_N^\beta f := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m}^\beta (\beta \text{qu})_{\nu,m}^*. \quad (97)$$

Let  $\lambda^\beta = \{\lambda_{\nu,m}^\beta\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$  and  $\epsilon > 0$  be sufficiently small such that  $0 < \epsilon < \rho - r$ . The right hand side of (97) is not a quarkonial decomposition. However we can regard  $2^{-(r+\epsilon)|\beta|} \text{Ext}_N^\beta f$  as an atomic decomposition by the family of smooth atoms with no moment condition. Hence we have

$$\left\| \text{Ext}_N^\beta f \right\|_{A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)} \lesssim 2^{(\rho+\epsilon)|\beta|} \left\| \lambda^\beta \right\|_{a_{p(\cdot),q(\cdot)}^{s(\cdot)}} \lesssim \frac{\|\lambda\|_{a_{p(\cdot),q(\cdot),\rho}^{s(\cdot)}}}{2^{\delta|\beta|}} \lesssim \frac{\|f\|_{A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)}}{2^{\delta|\beta|}}.$$

Therefore, we obtain

$$\left\| \text{Ext}_N f \right\|_{A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{A_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)}$$

by Lemma 2.1. This implies that  $\text{Ext}_N$  is a continuous mapping and has desired properties.

**Step 3.** In this step, we reduce the condition  $0 > \left(\frac{n}{p(\cdot)} - s(\cdot)\right)^+$ . We take a  $\sigma \in \mathbb{R}$  such that  $0 > \left(\frac{n}{p(\cdot)} - (s(\cdot) + \sigma)\right)^+$  and  $nN < -N + \sigma < N + \sigma$ . Let  $L \in \mathbb{N}$  be sufficiently large enough to satisfy  $L \gg 1$  and  $N + \sigma \leq L$ . We can construct  $\text{Ext}_L : A_{p(\cdot), q(\cdot)}^{s(\cdot) + \sigma}(\mathbb{R}_+^n) \longrightarrow A_{p(\cdot), q(\cdot)}^{s(\cdot) + \sigma}(\mathbb{R}^n)$  by using the same argument in Step 2. We rewrite  $\text{Ext}_L$  to  $E^*$ . Then, by Step 2, we see that  $E^*|_{A_{p(\cdot), q(\cdot)}^{s(\cdot) + \sigma}(\mathbb{R}_+^n)}$  is a continuous mapping from  $A_{p(\cdot), q(\cdot)}^{s(\cdot) + \sigma}(\mathbb{R}_+^n)$  to  $A_{p(\cdot), q(\cdot)}^{s(\cdot) + \sigma}(\mathbb{R}^n)$  and that  $E^*f|_{\mathbb{R}_+^n} = f$  holds for any  $f \in A_{p(\cdot), q(\cdot)}^{s(\cdot) + \sigma}(\mathbb{R}_+^n)$ . Then,  $J_{-\sigma}$  is the continuous mapping from  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$  to  $A_{p(\cdot), q(\cdot)}^{s(\cdot) + \sigma}(\mathbb{R}_+^n)$  and  $J_\sigma$  is also the continuous mapping from  $A_{p(\cdot), q(\cdot)}^{s(\cdot) + \sigma}(\mathbb{R}^n)$  to  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ . Therefore, following composite mapping

$$\text{Ext}_N := J_\sigma \circ E^* \circ J_{-\sigma} : A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n) \rightarrow A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n) \quad (98)$$

make a sense.

We will prove  $\text{Ext}_N f|_{\mathbb{R}_+^n} = f$ . Let  $\phi \in \mathcal{D}(\mathbb{R}_+^n)$  and  $g \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  such that  $f = g|_{\mathbb{R}_+^n}$ . Let  $E : C_c^\infty(\mathbb{R}_+^n) \rightarrow C^\infty(\mathbb{R}^n)$  be a zero extension operator. Then we see that

$$\begin{aligned} \langle \text{Ext}_N f|_{\mathbb{R}_+^n}, \phi \rangle &= \langle \text{Ext}_N f, E\phi \rangle \\ &= \langle E^* J_\sigma f, \mathcal{F} \left[ \varphi^{(\sigma)} \mathcal{F}^{-1} E\phi \right] \rangle \\ &= \langle J_{-\sigma} f, \mathcal{F} \left[ \varphi^{(\sigma)} \mathcal{F}^{-1} E\phi \right] |_{\mathbb{R}_+^n} \rangle \end{aligned}$$

by the properties of  $E^*$ . Hence we obtain

$$\langle \text{Ext}_N f|_{\mathbb{R}_+^n}, \phi \rangle = \langle J_{-\sigma} g, \mathcal{F} \left[ \varphi^{(\sigma)} \mathcal{F}^{-1} E\phi \right] \rangle = \langle g, E\phi \rangle = \langle f, \phi \rangle$$

by the definition of the extension operator  $\text{Ext}_N$ . Therefore, we have  $\text{Ext}_N f|_{\mathbb{R}_+^n} = f$  for any  $f \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$ .  $\square$

**6.3. Trace operator for  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$ .** We extend Theorem 5.1 to  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$ . We consider the Trace operator

$$\text{Tr}_{\mathbb{R}_+^n} : f(x', x_n) \in \mathcal{S}(\mathbb{R}_+^n) \longmapsto f(x', 0) \in \mathcal{S}(\mathbb{R}^{n-1}).$$

**Theorem 6.9.** Assume that  $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ .

(1) Let  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$  satisfy

$$\text{ess inf}_{x \in \mathbb{R}^n} \left\{ s(\cdot) - \left[ \frac{1}{p(\cdot)} + (n-1) \left( \frac{1}{\min(1, p(\cdot))} - 1 \right) \right] \right\} > 0.$$

(a) The operator  $\text{Tr}_{\mathbb{R}_+^n}$  can be extended as a surjective and continuous mapping from

$$B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n) \text{ to } B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1}).$$

(b) The operator  $\text{Tr}_{\mathbb{R}_+^n}$  can be extended as a surjective and continuous mapping from

$$F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n) \text{ to } F_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1}).$$

(2) Let  $s(\cdot) \in C^{\log}(\mathbb{R}^n)$  and  $k \in \mathbb{N}_0$  satisfy

$$\text{ess inf}_{x \in \mathbb{R}^n} \left\{ s(\cdot) - \left[ k + \frac{1}{p(\cdot)} + (n-1) \left( \frac{1}{\min(1, p(\cdot))} - 1 \right) \right] \right\} > 0.$$

- (a) If  $g_0 \in B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1})$ ,  $g_1 \in B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{p(\cdot)} - 1}(\mathbb{R}^{n-1})$ ,  $\dots$ ,  $g_k \in B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{p(\cdot)} - k}(\mathbb{R}^{n-1})$ , then there exists a  $f \in B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$  such that  $\text{Tr}_{\mathbb{R}_+^n}(f) = g_0$ ,  $\text{Tr}_{\mathbb{R}_+^n}(\partial_{x_n} f) = g_1$ ,  $\dots$ ,  $\text{Tr}_{\mathbb{R}_+^n}(\partial_{x_n}^k f) = g_k$ .
- (b) If  $g_0 \in F_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1})$ ,  $g_1 \in F_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{p(\cdot)} - 1}(\mathbb{R}^{n-1})$ ,  $\dots$ ,  $g_k \in F_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{p(\cdot)} - k}(\mathbb{R}^{n-1})$ , then there exists a  $f \in F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$  such that  $\text{Tr}_{\mathbb{R}_+^n}(f) = g_0$ ,  $\text{Tr}_{\mathbb{R}_+^n}(\partial_{x_n} f) = g_1$ ,  $\dots$ ,  $\text{Tr}_{\mathbb{R}_+^n}(\partial_{x_n}^k f) = g_k$ .

*Proof.* Let  $N$  be a sufficiency large. For  $f \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}_+^n)$ , we define  $\text{Tr}_{\mathbb{R}_+^n} f := \text{Tr}_{\mathbb{R}_+^n}[\text{Ext}_N f]$ , where  $\text{Ext}_N$  is the extension operator as in Theorem 6.8. Firstly, we will prove that

$$\text{Tr}_{\mathbb{R}_+^n}[f|_{\mathbb{R}_+^n}] = \lim_{\epsilon \downarrow 0} \text{Tr}_{\mathbb{R}^n} \text{Ext}_N[f(\cdot', \cdot_n + \epsilon)|_{\mathbb{R}_+^n}]. \quad (99)$$

By Theorem 4.17, we can expansion  $\text{Ext}_N f \in A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  so that

$$\text{Ext}_N f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta (\beta \text{qu})_{\nu, m}.$$

By using the same argument of the proof of Theorem 5.1, we have

$$\text{Tr}_{\mathbb{R}^n}[\text{Ext}_N f] = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta \text{Tr}_{\mathbb{R}^n}[(\beta \text{qu})_{\nu, m}]. \quad (100)$$

The partial summation  $\sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta \text{Tr}_{\mathbb{R}^n}[(\beta \text{qu})_{\nu, m}]$  in (100) is a bounded set of  $A_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ .

Hence there exists  $\kappa > 0$  and  $C > 0$  such that

$$\left\| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta \text{Tr}_{\mathbb{R}^n}[(\beta \text{qu})_{\nu, m}] \right\|_{A_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1})} \leq C 2^{-\kappa|\beta|},$$

where  $\kappa$  and  $C$  do not depend on  $\nu$  and  $\beta$ . Let  $\delta > 0$ . If we change  $s(\cdot)$  into  $s(\cdot) - \delta$ , then

$$\left\| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta \text{Tr}_{\mathbb{R}^n}[(\beta \text{qu})_{\nu, m}] \right\|_{A_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{p(\cdot)} - \delta}(\mathbb{R}^{n-1})} \leq C 2^{-\kappa|\beta| - \delta\nu}$$

holds by the definition of  $(\beta \text{qu})_{\nu, m}$ . Since  $A_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{p(\cdot)} - \delta}(\mathbb{R}^{n-1}) \hookrightarrow B_{\infty, \infty}^{\tilde{s}(\cdot) - \frac{n}{p(\cdot)} - \delta}(\mathbb{R}^{n-1})$  by Proposition 3.5, we have

$$\left\| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta \text{Tr}_{\mathbb{R}^n}[(\beta \text{qu})_{\nu, m}] \right\|_{B_{\infty, \infty}^{\tilde{s}(\cdot) - \frac{n}{p(\cdot)} - \delta}(\mathbb{R}^{n-1})} \lesssim 2^{-\kappa|\beta| - \delta\nu}. \quad (101)$$

Then we prove the limit

$$\text{Tr}_{\mathbb{R}^n}[\text{Ext}_N f] = \lim_{\epsilon \downarrow 0} \left( \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta \text{Tr}_{\mathbb{R}^n}[(\beta \text{qu})_{\nu, m}(\cdot', \cdot_n + \epsilon)] \right) \quad (102)$$

exists in  $\mathcal{S}'(\mathbb{R}^{n-1})$ . Let take  $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})$  arbitrary. Since

$$\left| \left\langle \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta \text{Tr}_{\mathbb{R}^n}[(\beta \text{qu})_{\nu, m}], \varphi \right\rangle \right| \lesssim 2^{-\kappa|\beta| - \delta\nu},$$

there exists a finitely set  $\mathcal{U} \subset \mathbb{N}_0^n \times \mathbb{N}$  for any  $\epsilon' > 0$  such that

$$\sum_{(\beta, \nu) \in \mathbb{N}_0^n \times \mathbb{N} \setminus \mathcal{U}} \left| \left\langle \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta \text{Tr}_{\mathbb{R}^n}[(\beta \text{qu})_{\nu, m}(\cdot', \cdot_n + \epsilon)], \varphi \right\rangle \right| \leq C \frac{1}{2} \epsilon', \quad (103)$$

where constant number  $C > 0$  does not depend on  $\epsilon \in (0, 1)$ . Let

$$A_j = \left\langle \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta \operatorname{Tr}_{\mathbb{R}^n}[(\beta \operatorname{qu})_{\nu, m}(\cdot', \cdot_n + \epsilon_j)], \varphi \right\rangle$$

for  $j = 1, 2$ , where  $0 < \epsilon_1, \epsilon_2 \ll 1$ . Since

$$\lim_{\epsilon \downarrow 0} \sum_{(\beta, \nu) \in \mathcal{U}} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu, m}^\beta \operatorname{Tr}_{\mathbb{R}^n}[(\beta \operatorname{qu})_{\nu, m}(\cdot', \cdot_n + \epsilon)]$$

exists, we see that

$$\sum_{(\beta, \nu) \in \mathcal{U}} |A_1 - A_2| < \frac{1}{2} \epsilon'. \quad (104)$$

By (103) and (104), the limit (102) exists in the sense of  $\mathcal{S}(\mathbb{R}^{n-1})$ . Hence we obtain

$$\operatorname{Tr}_{\mathbb{R}^n}[\operatorname{Ext}_N f] = \lim_{\epsilon \downarrow 0} \operatorname{Tr}_{\mathbb{R}^n}[\operatorname{Ext}_N f(\cdot', \cdot_n + \epsilon)]. \quad (105)$$

Therefore, we see that  $\operatorname{Tr}_{\mathbb{R}_+^n}$  does not depend on  $N$  because (99) and  $\operatorname{Tr}_{\mathbb{R}^n}[\operatorname{Ext}_N f(\cdot', \cdot_n + \epsilon)]$  does not depend on  $N$  for any  $\epsilon > 0$ .

Next, we prove that  $\operatorname{Tr}_{\mathbb{R}_+^n}$  is a surjection. By the surjectivity of the operator  $\operatorname{Tr}_{\mathbb{R}^n}$  on  $\mathbb{R}^n$ , we prove

$$\operatorname{Tr}_{\mathbb{R}^n} f = \begin{cases} \operatorname{Tr}_{\mathbb{R}_+^n}[f|_{\mathbb{R}_+^n}] \in B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1}) & \text{if } f \in B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot)}(\mathbb{R}^n), \\ \operatorname{Tr}_{\mathbb{R}_+^n}[f|_{\mathbb{R}_+^n}] \in F_{\tilde{p}(\cdot), \tilde{p}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1}) & \text{if } f \in F_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot)}(\mathbb{R}^n). \end{cases} \quad (106)$$

By using same argument of (105), we see that

$$\begin{cases} \operatorname{Tr}_{\mathbb{R}^n} f = \lim_{\epsilon \downarrow 0} \operatorname{Tr}_{\mathbb{R}^n} f(\cdot', \cdot_n + \epsilon), \\ \operatorname{Tr}_{\mathbb{R}_+^n}[f|_{\mathbb{R}_+^n}] = \lim_{\epsilon \downarrow 0} \operatorname{Tr}_{\mathbb{R}^n} \operatorname{Ext}_N[f(\cdot', \cdot_n + \epsilon)|_{\mathbb{R}_+^n}] \end{cases} \quad (107)$$

in  $\mathcal{S}'(\mathbb{R}^{n-1})$ . Here we have  $f(\cdot', \cdot_n + \epsilon) = \operatorname{Ext}_N[f(\cdot', \cdot_n + \epsilon)|_{\mathbb{R}_+^n}]$  on  $\{x \in \mathbb{R}^n : x_n \geq -\epsilon/2\}$ . Hence we obtain

$$\operatorname{Tr}_{\mathbb{R}^n} f = \lim_{\epsilon \downarrow 0} \operatorname{Tr}_{\mathbb{R}^n} f(\cdot', \cdot_n + \epsilon) = \lim_{\epsilon \downarrow 0} \operatorname{Tr}_{\mathbb{R}^n} \operatorname{Ext}_N[f(\cdot', \cdot_n + \epsilon)|_{\mathbb{R}_+^n}] = \operatorname{Tr}_{\mathbb{R}_+^n}[f|_{\mathbb{R}_+^n}].$$

This complete the proof of (1).

The assertion (2) follows from (1) and Theorem 5.1.  $\square$

#### ACKNOWLEDGEMENT

I would like to express my gratitude to Professor Yoshikazu Kobayashi for great support in any manner and for his valuable suggestions in many discussions. I also would like to express my gratitude to Professor Yoshihiro Sawano for sending his book [22]. I have obtained many ideas from the book [22].

#### REFERENCES

- [1] A. Almeida and P. Hästö, *Besov spaces with variable smoothness and integrability*. J. Funct. Anal. 258, no. 5 (2010) 1628–1655.
- [2] D. Cruz-Uribe, SFO, A. Fiorenza, J. M. Martell and C. Pérez, *The boundedness of classical operators on variable  $L^p$  spaces*. Ann. Acad. Sci. Fenn., Math. 31 (2006) 239–264.
- [3] L. Diening, *Maximal function on Musielak–Orlicz spaces and generalized lebesgue spaces* Bull. Sci. Math. 129 (2005) 657–700.
- [4] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Legesue and Sobolev Spaces with Variable Exponents*, Springer, Lecture Notes in Mathematics **2017**, Springer (2011).
- [5] L. Diening, P. Hästö and S. Roudenko, *Function spaces of variable smoothness and integrability*. J. Funct. Anal. 256 (2009) 1731–1768.

- [6] D. Drihem, *Atomic decomposition of Besov spaces with variable smoothness and integrability*. J. Math. Anal. Appl. 389 (2012) 15–31.
- [7] M. Fraizer and B. Jawerth, *Decomposition of Besov spaces*. Indiana Univ. Math. J. 34 (1985) 777–799.
- [8] M. Fraizer and B. Jawerth, *A discrete transform and decompositions of distribution spaces*. J. Funct. Anal. 93, 34–170 (1990).
- [9] J. Franke and T. Runst, *Regular elliptic boundary value problems in Besov-Triebel-Lizorkin space*. Math. Nachr. **174** (1995), 113–149.
- [10] A. Huang and J. Xu, *Multilinear singular integrals and commutators in variable exponent Lebesgue spaces*. Appl. Math. J. Chinese Univ. 2010, 25(1), 69–77.
- [11] H. Kempka, *2-microlocal Besov and Triebel-Lizorkin spaces of variable integrability*. Rev. Mat. Complut. **22** (2009), 227–251.
- [12] H. Kempka, *Atomic, molecular and wavelet decomposition of 2-microlocal Besov and Triebel-Lizorkin spaces with variable integrability*. Funct. Approx. Comment. Math. **43**, No. 2 (2010), 171–208.
- [13] H. Kempka and Jan Vybřál *A note on the spaces of variable integrability and summability of Almeida and Hästö*. arXiv. 1102.1597v1
- [14] O. Kováčik and J. Rákosník, *On spaces  $L^{p(x)}$  and  $W^{k,p(x)}$* . Czech. Math. J. 41 (1991) 592–618.
- [15] S. Moura, J. Neves and C. Schneider, *On trace spaces of 2-microlocal Besov spaces with variable integrability*. Math. Nachr., 286 (2013) 1240–1254.
- [16] T. Noi, *Fourier multiplier theorems for Besov and Triebel-Lizorkin spaces with variable exponents*. Math. Inequal. Appl. 17 (2014) 49–74.
- [17] T. Noi and M. Izuki, *Duality of Besov, Triebel-Lizorkin and Herz spaces with variable exponents*. to appear in Rend. Circ. Mat. Palermo.
- [18] T. Noi and Y. Sawano, *Complex interpolation of Besov spaces and Triebel-Lizorkin spaces with variable exponents*. J. Math. Anal. Appl. 387 (2012) 676–690.
- [19] W. Orlicz. *Über konjugierte Exponentenfolgen*. Studia Math., **3** (1931) 200–211.
- [20] L. Pick and M. Růžička, *An example of a space  $L^{p(\cdot)}$  on which the Hardy-Littlewood maximal operator is not bounded*. Expo. Math. 19 (2001) 369–371.
- [21] K. Rajagopal, M. Růžička, *On the modeling of electrorheological materials*. Mech. Res. Comm. **23** (1996) 401–407.
- [22] Y. Sawano, *Theory of Besov spaces*. (in Japanese) Nihon Hyo-ronsya, 2011.
- [23] H. Triebel, *Theory of Function Spaces*. Birkhäuser, Basel, Boston, 1983.
- [24] H. Triebel, *The Structure of Functions*. Birkhäuser, Basel, Boston, 2001.

Takahiro Noi

Department of Mathematics and Information Science, Tokyo Metropolitan University

1-1 Minami osawa, hachioji-city, Tokyo.

E-mail : taka.noi.hiro@gmail.com